

NICHOLS ALGEBRAS OF UNIDENTIFIED DIAGONAL TYPE

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ABSTRACT. The Nichols algebras of diagonal type with finite root system are either of standard, super or (yet) unidentified type. A concrete description of the defining relations of all those Nichols algebras was given in [A3]. In the present paper we use this result to give an explicit presentation of all Nichols algebras of unidentified type.

INTRODUCTION

The classification of braidings of diagonal type whose Nichols algebra has a finite number of roots was given in [H2]. This problem is related with the classification of finite-dimensional pointed Hopf algebras over abelian groups. The list of Heckenberger can be split off in three families:

- standard braidings, introduced in [AA];
- braidings of super type, see [AAY];
- a finite list of braidings whose connected components have rank less than eight.

We call them *unidentified*.

We recall that the Nichols algebra of a braided vector space (V, c) is a quotient of its tensor algebra by a suitable ideal $I(V)$. A crucial question involving these Nichols algebras is to obtain a minimal set of relations generating $I(V)$. Such problem was solved in [A1] for the first family, with a formula for the dimension of each Nichols algebra of standard type. A presentation for the second family was given in [Y] for the generic case, and in [AAY] for the non-generic case (except by some considerations for small orders on the entries of the braiding matrix).

A complete answer can be found in [A3], where the main result gives a list of relations satisfied by the generators of the Nichols algebras, depending on the matrix entries, see Theorem 1.6 below. This paper depends strongly on [A2], the key point to obtain the desired presentation. It remains the problem to identify the relations needed for each one of the braidings. This is instrumental for several important questions concerning pointed Hopf algebras, among them the explicit determination of all liftings and their representation theory.

In this paper, we deal with unidentified braidings and give a complete list of relations generating the defining ideal for the Nichols algebra of each braiding of this kind. For the small ranks we give also the list of positive roots for each case and the dimension.

The organization of the paper is the following. The first Section includes some notions about PBW bases and root systems of Nichols algebras of diagonal type, as well as the presentation by generators and relations from [A3]. The second Section is devoted to give explicitly the presentation for each diagram. We begin with some unidentified families in rank 2, 3 or 4, and finally we split the remaining braidings in families by considering certain similarities on the associated generalized Cartan matrices.

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Notation. \mathbf{k} will denote an algebraically closed field of characteristic zero. For each $N > 1$, \mathbb{G}_N will denote the group of N -roots of unity in \mathbf{k} , and \mathbb{G}'_N the corresponding subset of primitive roots of order N .

1. PRELIMINARIES

We will recall all the preliminary results and fix the notation that we will use along this work. They are related with the theory of PBW bases for braided Hopf algebras of diagonal type [Kh], the Weyl groupoid of diagonal braided vector spaces [H1] and the presentation of Nichols algebras by generators and relations [A3].

1.1. PBW bases for Nichols algebras of diagonal type.

We begin with the definition of a Nichols algebra associated to a braided vector space. To this end, fix a Hopf algebra H with bijective antipode, and denote by ${}^H_H\mathcal{YD}$ the category of left Yetter-Drinfeld modules over H . Let $V \in {}^H_H\mathcal{YD}$. The tensor algebra $T(V)$ admits a braiding extending $c : V \otimes V \rightarrow V \otimes V$, and under such braiding it has a unique structure of graded braided Hopf algebra in ${}^H_H\mathcal{YD}$ such that $V \subseteq \mathcal{P}(V)$ (i.e. $\Delta(x) = x \otimes 1 + 1 \otimes x$).

Definition 1.1. [AS3] Let \mathfrak{S} be the family of all the homogeneous Hopf ideals of $I \subseteq T(V)$ such that

- I is generated by homogeneous elements of degree ≥ 2 ,
- I is a Yetter-Drinfeld submodule of $T(V)$.

The *Nichols algebra* $\mathcal{B}(V)$ associated to V is the quotient of $T(V)$ by the biggest ideal $I(V)$ of \mathfrak{S} .

The definition does not depend on the realization of a braided vector space (V, c) as a Yetter-Drinfeld module. In particular we will consider braidings of diagonal type; that is, there exists a basis $\{x_i\}_{i \in I}$ of V and a family of non-zero scalars $(q_{ij})_{i,j \in I}$ such that $c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i$. These braided vector spaces are related with vector spaces over group algebras of finite abelian groups.

Fix (V, c) a braided vector space of diagonal type. We will describe, following [Kh], a particular PBW basis for each graded braided Hopf algebra $\mathcal{B} = \bigoplus_{n \in \mathbb{N}} \mathcal{B}^n$ generated by $\mathcal{B}^1 \cong V$ as an algebra.

Assume that V is finite dimensional, and denote $\theta = \dim V$. Fix a basis $X = \{x_1, \dots, x_\theta\}$ of V as above. \mathbb{X} will denote the set of words with letters in X . We consider the lexicographical order on \mathbb{X} . We can identify $\mathbf{k}\mathbb{X}$ with $T(V)$.

$T(V)$ admits a unique \mathbb{Z}^I -graduation as a braided Hopf algebra such that $\deg x_i = \alpha_i$, where $(\alpha_i)_{i \in I}$ is the canonical basis of \mathbb{Z}^I . Assume that $\dim V = \theta < \infty$. Let $\chi : \mathbb{Z}^\theta \times \mathbb{Z}^\theta \rightarrow \mathbf{k}^\times$ be the bicharacter determined by the condition

$$(1) \quad \chi(\alpha_i, \alpha_j) = q_{ij}, \quad \text{for each pair } 1 \leq i, j \leq \theta.$$

Then, for each pair of \mathbb{Z}^θ -homogeneous elements $u, v \in \mathbb{X}$,

$$(2) \quad c(u \otimes v) = q_{u,v} v \otimes u, \quad q_{u,v} = \chi(\deg u, \deg v) \in \mathbf{k}^\times.$$

Note that [AS3, Prop. 2.10] implies that $I(V)$ is a \mathbb{Z}^θ -homogeneous ideal, and then $\mathcal{B}(V)$ is \mathbb{Z}^θ -graded, see also [L, Prop. 1.2.3].

Definition 1.2. $u \in \mathbb{X} - \{1\}$ is a *Lyndon word* if for every decomposition $u = vw$, $v, w \in \mathbb{X} - \{1\}$, it holds that $u < w$. We will denote the set of all Lyndon words by L .

We know that any word $u \in \mathbb{X}$ admits a unique decomposition as a non-increasing product of Lyndon words:

$$(3) \quad u = l_1 l_2 \dots l_r, \quad l_i \in L, l_r \leq \dots \leq l_1.$$

It is called the *Lyndon decomposition* of $u \in \mathbb{X}$, and each $l_i \in L$ in (3) is called a *Lyndon letter* of u .

For each $u \in L - X$, the *Shirshov decomposition* of u is the decomposition $u = u_1 u_2$, $u_1, u_2 \in L$, such that u_2 is the smallest end of u between all the possible decompositions satisfying these conditions (it is easily proved that each Lyndon word admits at least one of such decompositions).

For a general braided vector space, the *braided bracket* of $x, y \in T(V)$ is defined by

$$(4) \quad [x, y]_c := \text{multiplication} \circ (\text{id} - c)(x \otimes y).$$

Using the previous decompositions and the identification of \mathbb{X} with a basis of $T(V)$, we can define a \mathbf{k} -linear endomorphism $[-]_c$ of $T(V)$ as follows:

$$[u]_c := \begin{cases} u, & \text{if } u = 1 \text{ or } u \in X; \\ [[v]_c, [w]_c]_c, & \text{if } u \in L, \ell(u) > 1, u = vw \text{ is the Shirshov decomposition}; \\ [u_1]_c \dots [u_t]_c, & \text{if } u \in \mathbb{X} - L \text{ and its Lyndon decomposition is } u = u_1 \dots u_t. \end{cases}$$

We will obtain PBW bases using this automorphism.

Definition 1.3. The *hyperletter* corresponding to $l \in L$ is the element $[l]_c$. An *hyperword* is a word written in hyperletters; a *monotone hyperword* is an hyperword $W = [u_1]_c^{k_1} \dots [u_m]_c^{k_m}$ such that $u_1 > \dots > u_m$.

A different order on \mathbb{X} , considered in [U] and used implicitly in [Kh] is the *deg-lex order*, defined as follows. For each pair $u, v \in \mathbb{X}$, we say that $u \succ v$ if $\ell(u) < \ell(v)$, or $\ell(u) = \ell(v)$ and $u > v$ for the lexicographical order. Such order is total. The empty word 1 is the maximal element for \succ , and this order is invariant by left and right multiplication.

In what follows I will denote a Hopf ideal, and $R = T(V)/I$. Let $\pi : T(V) \rightarrow R$ be the canonical projection. We set:

$$G_I := \{u \in \mathbb{X} : u \notin \mathbf{k}\mathbb{X}_{\succ u} + I\}.$$

Therefore, if $u \in G_I$ and $u = vw$, then $v, w \in G_I$. In this way, each $u \in G_I$ is a non-increasing product of Lyndon words of G_I .

Consider the set $S_I := G_I \cap L$, and define $h_I : S_I \rightarrow \{2, 3, \dots\} \cup \{\infty\}$ by the condition:

$$(5) \quad h_I(u) := \min \{t \in \mathbb{N} : u^t \in \mathbf{k}\mathbb{X}_{\succ u^t} + I\}.$$

Following [Kh] we have the following results.

Theorem 1.4. *The set*

$$\{[u_1]_c^{k_1} \dots [u_m]_c^{k_m} : m \in \mathbb{N}_0, u_1 > \dots > u_m, u_i \in S_I, 0 < k_i < h_I(u_i)\}$$

is a PBW basis of $H = T(V)/I$. □

Corollary 1.5. (i) *A word u belongs to G_I if and only if the hyperletter $[u]_c$ is not a linear combination of greater hyperwords $[w]_c$, $w \succ u$, whose hyperletters are in S_I , modulo I .* □

(ii) *If $v \in S_I$ is such that $h_I(v) < \infty$, then $q_{v,v}$ is a root of unity. Moreover, if $\text{ord } q_{v,v} = h$, then $h_I(v) = h$, and $[v]^h$ is a linear combination of hyperwords $[w]_c$, $w \succ v^h$.* □

Let Δ_+^V be the set of degrees of a PBW basis of $\mathcal{B}(V)$, counted with their multiplicities [H1]. We can see that it does not depend on the PBW basis, [H1, AA]. We can attach a Cartan scheme \mathcal{C} , a Weyl groupoid \mathcal{W} and a root system \mathcal{R} in the sense of [CH, HY], see [HS, Thms. 6.2, 6.9]. To this end, define for each $1 \leq i \neq j \leq \theta$,

$$(6) \quad -a_{ij} := \min \{n \in \mathbb{N}_0 : (n+1)_{q_{ii}}(1 - q_{ii}^n \widetilde{q_{ij}}) = 0\},$$

and set $a_{ii} = 2$. The symmetry $s_i \in \text{Aut}(\mathbb{Z}^\theta)$ is defined by the condition $s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i$.

Set $\widetilde{q}_{rs} = \chi(s_i(\alpha_r), s_i(\alpha_s))$. Let V_i be another vector space of the same dimension, and attach to it the matrix $\widetilde{\mathbf{q}} = (\widetilde{q}_{rs})$. By [H1],

$$\Delta_+^{V_i} = s_i(\Delta_+^V \setminus \{\alpha_i\}) \cup \{\alpha_i\}.$$

Therefore last equation lets us to define the Weyl groupoid of V , whose root system is defined by the sets $\Delta^{V'}$, V' obtained after to apply some reflections to the matrix of V . Those braided vector spaces obtained after to apply the symmetries s_i define the *Weyl equivalence class* of V .

When the root system is finite, we can prove that each root is real, and in consequence it has multiplicity one, see [CH].

1.2. A presentation by generators and relations of Nichols algebras of diagonal type.

Fix as above a finite-dimensional braided vector space (V, c) of diagonal type, with braiding matrix $(q_{ij})_{1 \leq i, j \leq \theta}$, $\theta = \dim V$, and a basis x_1, \dots, x_θ of V such that $c(x_i \otimes x_j) = q_{ij}x_j \otimes x_i$. Let χ be the bicharacter associated to (q_{ij}) .

We denote $\widetilde{q}_{ij} = q_{ij}q_{ji}$. The *generalized Dynkin diagram* of a matrix $(q_{ij})_{1 \leq i, j \leq \theta}$ is a graph with θ vertices, labeled with the scalars q_{ii} , and an arrow between the vertices i and j if $\widetilde{q}_{ij} \neq 1$, labeled with this scalar. For example, given $q \in \mathbf{k}^\times$, the matrices

$$\begin{pmatrix} q^2 & q^{-2} & 1 \\ 1 & q^2 & q^{-2} \\ 1 & 1 & q \end{pmatrix}, \begin{pmatrix} q^2 & q^{-1} & q \\ q^{-1} & q^2 & q^{-1} \\ q^{-1} & q^{-1} & q \end{pmatrix}$$

have the diagram: $\circ_{q^2} \xrightarrow{q^{-2}} \circ_{q^2} \xrightarrow{q^{-2}} \circ_q$.

In fact, two braided vector spaces of diagonal type are *twist equivalent* [AS3] if they have the same generalized Dynkin diagram.

We denote also

$$x_{i_1 i_2 \dots i_k} = (\text{ad}_c x_{i_1}) \cdots (\text{ad}_c x_{i_{k-1}}) x_{i_k}, \quad i_j \in \{1, \dots, \theta\}.$$

For each $m \in \mathbb{N}$, we define the elements $x_{(m+1)\alpha_i + m\alpha_j} \in \mathcal{U}(\chi)$ recursively:

- if $m = 1$, $x_{2\alpha_i + \alpha_j} := (\text{ad}_c x_i)^2 x_j = x_{ii} x_j$,
- $x_{(m+2)\alpha_i + (m+1)\alpha_j} := [x_{(m+1)\alpha_i + m\alpha_j}, x_{ij}]_c$.

Call x_α , $\alpha \in \Delta_+^V$, the generator of the Kharchenko's PBW basis. We denote

$$N_\alpha := \text{ord } \chi(\alpha, \alpha), \quad \text{if } \chi(\alpha, \alpha) \text{ is a root of unity.}$$

We give now the main result of [A3].

Theorem 1.6. *Assume that the root system Δ^χ is finite. Then $\mathcal{B}(V)$ admits a presentation by generators x_1, \dots, x_θ and relations:*

$$(7) \quad x_\alpha^{N_\alpha}, \quad \alpha \in \mathcal{O}(\chi);$$

$$(8) \quad (\text{ad}_c x_i)^{m_{ij}+1} x_j, \quad q_{ii}^{m_{ij}+1} \neq 1;$$

$$(9) \quad x_i^{N_i}, \quad i \text{ is not a Cartan vertex};$$

⊙ if $i, j \in \{1, \dots, \theta\}$ are such that $q_{ii} = \widetilde{q}_{ij} = q_{jj} = -1$, and there exists $k \neq i, j$ such that $\widetilde{q}_{ik}^2 \neq 1$ or $\widetilde{q}_{jk}^2 \neq 1$,

$$(10) \quad x_{ij}^2;$$

⊙ if $i, j, k \in \{1, \dots, \theta\}$ are such that $q_{jj} = -1$, $\widetilde{q}_{ik} = \widetilde{q}_{ij}\widetilde{q}_{kj} = 1$,

$$(11) \quad [x_{ijk}, x_j]_c;$$

⊙ if $i, j \in \{1, \dots, \theta\}$ are such that $q_{jj} = -1$, $q_{ii}\widetilde{q}_{ij} \in \mathbb{G}'_6$, and also $q_{ii} \in \mathbb{G}'_3$ or $m_{ij} \geq 3$,

$$(12) \quad [x_{iij}, x_{ij}]_c;$$

⊙ if $i, j, k \in \{1, \dots, \theta\}$ are such that $q_{ii} = \pm \widetilde{q}_{ij} \in \mathbb{G}'_3$, $\widetilde{q}_{ik} = 1$, and also $-q_{jj} = \widetilde{q}_{ij}\widetilde{q}_{jk} = 1$ or $q_{jj}^{-1} = \widetilde{q}_{ij} = \widetilde{q}_{jk} \neq -1$,

$$(13) \quad [x_{iijk}, x_{ij}]_c;$$

⊙ if $i, j, k \in \{1, \dots, \theta\}$ are such that $\widetilde{q}_{ik}, \widetilde{q}_{ij}, \widetilde{q}_{jk} \neq 1$,

$$(14) \quad x_{ijk} - \frac{1 - \widetilde{q}_{jk}}{q_{kj}(1 - \widetilde{q}_{ik})} [x_{ik}, x_j]_c - q_{ij}(1 - \widetilde{q}_{jk}) x_j x_{ik};$$

⊙ if $i, j, k \in \{1, \dots, \theta\}$ are such that one of the following situations

- ∘ $q_{ii} = q_{jj} = -1$, $\widetilde{q}_{ij}^2 = \widetilde{q}_{jk}^{-1}$, $\widetilde{q}_{ik} = 1$, or
- ∘ $\widetilde{q}_{ij} = q_{jj} = -1$, $q_{ii} = -\widetilde{q}_{jk}^2 \in \mathbb{G}'_3$, $\widetilde{q}_{ik} = 1$, or
- ∘ $q_{kk} = \widetilde{q}_{jk} = q_{jj} = -1$, $q_{ii} = -\widetilde{q}_{ij} \in \mathbb{G}'_3$, $\widetilde{q}_{ik} = 1$, or
- ∘ $q_{jj} = -1$, $\widetilde{q}_{ij} = q_{ii}^{-2}$, $\widetilde{q}_{jk} = -q_{ii}^{-3}$, $\widetilde{q}_{ik} = 1$, or
- ∘ $q_{ii} = q_{jj} = q_{kk} = -1$, $\pm \widetilde{q}_{ij} = \widetilde{q}_{jk} \in \mathbb{G}'_3$, $\widetilde{q}_{ik} = 1$,

$$(15) \quad [[x_{ij}, x_{ijk}]_c, x_j]_c;$$

⊙ if $i, j, k \in \{1, \dots, \theta\}$ are such that $q_{ii} = q_{jj} = -1$, $\widetilde{q}_{ij}^3 = \widetilde{q}_{jk}^{-1}$, $\widetilde{q}_{ik} = 1$,

$$(16) \quad [[x_{ij}, [x_{ij}, x_{ijk}]_c], x_j]_c;$$

⊙ if $i, j, k, l \in \{1, \dots, \theta\}$ are such that $q_{jj}\widetilde{q}_{ij} = q_{jj}\widetilde{q}_{jk} = 1$, $\widetilde{q}_{jk}^2 = \widetilde{q}_{lk}^{-1} = q_{ll}$, $q_{kk} = -1$, $\widetilde{q}_{ik} = \widetilde{q}_{il} = \widetilde{q}_{jl} = 1$,

$$(17) \quad [[[x_{ijkl}, x_k]_c, x_j]_c, x_k]_c;$$

⊙ if $i, j, k \in \{1, \dots, \theta\}$ are such that $q_{jj} = \widetilde{q}_{ij}^2 = \widetilde{q}_{jk} \in \mathbb{G}'_3$, $\widetilde{q}_{ik} = 1$,

$$(18) \quad [[x_{ijk}, x_j]_c, x_j]_c;$$

⊙ if $i, j, k \in \{1, \dots, \theta\}$ are such that $q_{jj} = \widetilde{q}_{ij}^3 = \widetilde{q}_{jk} \in \mathbb{G}'_4$, $\widetilde{q}_{ik} = 1$,

$$(19) \quad [[[x_{ijk}, x_j]_c, x_j]_c, x_j]_c;$$

⊙ if $i, j, k \in \{1, \dots, \theta\}$ are such that $q_{ii} = \widetilde{q}_{ij} = -1$, $q_{jj} = \widetilde{q}_{jk}^{-1} \neq -1$, $\widetilde{q}_{ik} = 1$,

$$(20) \quad [x_{ij}, x_{ijk}]_c;$$

⊙ if $i, j, k \in \{1, \dots, \theta\}$ are such that $q_{ii} = q_{kk} = -1$, $\widetilde{q}_{ik} = 1$, $\widetilde{q}_{ij} \in \mathbb{G}'_3$, $q_{jj} = -\widetilde{q}_{jk} = \pm \widetilde{q}_{ij}$,

$$(21) \quad [x_i, x_{jjk}]_c - (1 + q_{jj}^2)q_{kj}^{-1} [x_{ijk}, x_j]_c - (1 + q_{jj}^2)(1 + q_{jj})q_{ij}x_jx_{ijk};$$

⊙ if $i, j, k \in \{1, \dots, \theta\}$ are such that $\widetilde{q}_{jk} = 1$, $q_{ii} = \widetilde{q}_{ij} = -\widetilde{q}_{ik} \in \mathbb{G}'_3$,

$$(22) \quad [x_i, [x_{ij}, x_{ik}]_c]_c + q_{jk}q_{ik}q_{ji} [x_{iik}, x_{ij}]_c + q_{ij}x_{ij}x_{iik};$$

⊙ if $i, j, k \in \{1, \dots, \theta\}$ are such that $q_{jj} = q_{kk} = \widetilde{q}_{jk} = -1$, $q_{ii} = -\widetilde{q}_{ij} \in \mathbb{G}'_3$, $\widetilde{q}_{ik} = 1$,

$$(23) \quad [x_{iijk}, x_{ijk}]_c;$$

⊙ if $i, j \in \{1, \dots, \theta\}$ are such that $-q_{ii}, -q_{jj}, q_{ii}\widetilde{q_{ij}}, q_{jj}\widetilde{q_{ij}} \neq 1$,

$$(24) \quad (1 - \widetilde{q_{ij}})q_{jj}q_{ji} [x_i, [x_{ij}, x_j]_c]_c - (1 + q_{jj})(1 - q_{jj}\widetilde{q_{ij}})x_{ij}^2;$$

⊙ if $i, j \in \{1, \dots, \theta\}$ are such that either $m_{ij} \in \{4, 5\}$, or else $q_{jj} = -1$, $m_{ij} = 3$ and $q_{ii} \in \mathbb{G}'_4$,

$$(25) \quad [x_i, x_{3\alpha_i+2\alpha_j}]_c - \frac{1 - q_{ii}\widetilde{q_{ij}} - q_{ii}^2\widetilde{q_{ij}}^2q_{jj}}{(1 - q_{ii}\widetilde{q_{ij}})q_{ji}}x_{ij}^2;$$

⊙ if $i, j \in \{1, \dots, \theta\}$ are such that $4\alpha_i + 3\alpha_j \notin \Delta_+^\chi$, $q_{jj} = -1$ or $m_{ji} \geq 2$, and also $m_{ij} \geq 3$, or $m_{ij} = 2$, $q_{ii} \in \mathbb{G}'_3$,

$$(26) \quad x_{4\alpha_i+3\alpha_j} = [x_{3\alpha_i+2\alpha_j}, x_{ij}]_c;$$

⊙ if $i, j \in \{1, \dots, \theta\}$ are such that $3\alpha_i + 2\alpha_j \in \Delta_+^\chi$, $5\alpha_i + 3\alpha_j \notin \Delta_+^\chi$, and $q_{ii}^3\widetilde{q_{ij}}, q_{ii}^4\widetilde{q_{ij}} \neq 1$,

$$(27) \quad [x_{ij}, x_{3\alpha_i+2\alpha_j}]_c;$$

⊙ if $i, j \in \{1, \dots, \theta\}$ are such that $4\alpha_i + 3\alpha_j \in \Delta_+^\chi$, $5\alpha_i + 4\alpha_j \notin \Delta_+^\chi$,

$$(28) \quad x_{5\alpha_i+4\alpha_j} = [x_{4\alpha_i+3\alpha_j}, x_{ij}]_c;$$

⊙ if $i, j \in \{1, \dots, \theta\}$ are such that $5\alpha_i + 2\alpha_j \in \Delta_+^\chi$, $7\alpha_i + 3\alpha_j \notin \Delta_+^\chi$,

$$(29) \quad [[x_{ij}, x_{ij}], x_{ij}]_c;$$

⊙ if $i, j \in \{1, \dots, \theta\}$ are such that $q_{jj} = -1$, $5\alpha_i + 4\alpha_j \in \Delta_+^\chi$,

$$(30) \quad [x_{ij}, x_{4\alpha_i+3\alpha_j}]_c - \frac{b - (1 + q_{ii})(1 - q_{ii}\zeta)(1 + \zeta + q_{ii}\zeta^2)q_{ii}^6\zeta^4}{a q_{ii}^3q_{ij}^2q_{ji}^3}x_{3\alpha_i+2\alpha_j}^2,$$

where $\zeta = \widetilde{q_{ij}}$, $a = (1 - \zeta)(1 - q_{ii}^4\zeta^3) - (1 - q_{ii}\zeta)(1 + q_{ii})q_{ii}\zeta$, $b = (1 - \zeta)(1 - q_{ii}^6\zeta^5) - a q_{ii}\zeta$. \square

2. UNIDENTIFIED NICHOLS ALGEBRAS OF RANK 2

In the following Sections we consider the different Weyl equivalence classes of braided vector spaces of unidentified type. We divide the work depending on the dimension of such spaces. We consider in this Section the 2-dimensional unidentified spaces, then some particular cases in rank three and four, and finally the remaining cases, but dividing the work in some families, according with the shape of the associated generalized Dynkin diagram.

The unidentified braided vector spaces in [H2, Table 1] of rank two are those in rows 7, 8, 9, 12, 13, 14, 15 and 16.

We will consider each possible row, describe the root system of each braiding, and calculate the dimension of the corresponding Nichols algebra.

Remark 2.1. The hyperword associated to a simple root α_i is the one associated to the unique Lyndon word of degree α_i : x_i . Also, the hyperword associated to a root of the way $m\alpha_1 + \alpha_2$ is $x_{m\alpha_1+\alpha_2} = (\text{ad}_c x_1)^m x_2$.

Moreover, for the braidings considered in this Subsection we have the following possible hyperwords:

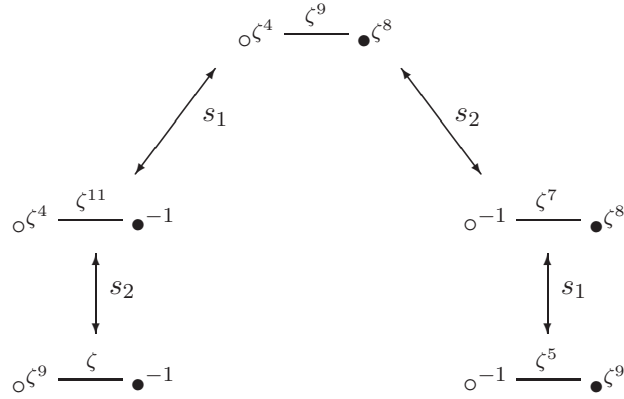
$$\begin{aligned}
x_{\alpha_1+2\alpha_2} &= [x_{\alpha_1+\alpha_2}, x_2]_c, & x_{3\alpha_1+2\alpha_2} &= [x_{2\alpha_1+\alpha_2}, x_{\alpha_1+\alpha_2}]_c, \\
x_{4\alpha_1+3\alpha_2} &= [x_{3\alpha_1+2\alpha_2}, x_{\alpha_1+\alpha_2}]_c, & x_{5\alpha_1+2\alpha_2} &= [x_{3\alpha_1+\alpha_2}, x_{2\alpha_1+\alpha_2}]_c, \\
x_{5\alpha_1+3\alpha_2} &= [x_{2\alpha_1+\alpha_2}, x_{3\alpha_1+2\alpha_2}]_c, & x_{5\alpha_1+4\alpha_2} &= [x_{4\alpha_1+3\alpha_2}, x_{\alpha_1+\alpha_2}]_c, \\
x_{7\alpha_1+2\alpha_2} &= [x_{4\alpha_1+\alpha_2}, x_{3\alpha_1+\alpha_2}]_c, & x_{7\alpha_1+3\alpha_2} &= [x_{5\alpha_1+2\alpha_2}, x_{2\alpha_1+\alpha_2}]_c, \\
x_{7\alpha_1+4\alpha_2} &= [x_{2\alpha_1+\alpha_2}, x_{5\alpha_1+3\alpha_2}]_c, & x_{8\alpha_1+3\alpha_2} &= [x_{3\alpha_1+\alpha_2}, x_{5\alpha_1+2\alpha_2}]_c, \\
x_{8\alpha_1+5\alpha_2} &= [x_{5\alpha_1+3\alpha_2}, x_{3\alpha_1+2\alpha_2}]_c.
\end{aligned}$$

Remark 2.2. If V, W are two Weyl equivalent braided vector spaces of diagonal type, then $\dim \mathcal{B}(V) = \dim \mathcal{B}(W)$. It follows from the fact

$$\Delta_+^{s_p^* \chi} = s_p(\Delta^\chi \setminus \{\alpha_p\}) \cup \{\alpha_p\}$$

and that a hyperword of degree α has height $\text{ord } \chi(\alpha, \alpha)$, so we calculate the dimension computing the number of terms of the PBW basis; i.e. multiplying the orders of the associated scalars.

Example 2.3. Row 7. These are the first unidentified braided vector spaces, for which $\zeta \in \mathbb{G}'_{12}$ is primitive. The following diagram shows the action of the Weyl groupoid, where we indicate the vertices 1, 2 by \circ, \bullet , respectively. We omit those symmetries not changing the Dynkin diagram.



In this case, $\mathcal{O}(\chi)$ is empty. For each one of these vector spaces V , $\dim \mathcal{B}(V) = 2^4 3^2 = 144$.

(i) $\circ \zeta^4 \xrightarrow{\zeta^9} \bullet \zeta^8$. In this case,

$$\Delta_+^\chi = \{\alpha_1, 2\alpha_1 + \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, \alpha_2\}.$$

Considering the corresponding hyperwords from Remark 2.1 and following Theorem 1.6, $\mathcal{B}(V)$ has a presentation by generators x_1, x_2 , and relations

$$x_1^3 = x_2^3 = [x_1, x_{\alpha_1+2\alpha_2}]_c - \frac{\zeta^{10}(1-\zeta^7)q_{12}}{1-\zeta^9} x_{\alpha_1+\alpha_2}^2 = 0.$$

(ii) $\circ \zeta^4 \xrightarrow{\zeta^{11}} \bullet^{-1}$. The set of positive roots is in this case

$$\Delta_+^\chi = \{\alpha_1, 2\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2, \alpha_1 + \alpha_2, \alpha_2\}.$$

Therefore $\mathcal{B}(V)$ has a presentation by generators x_1, x_2 , and relations

$$x_1^3 = x_2^2 = [x_{3\alpha_1+2\alpha_2}, x_{12}]_c = 0.$$

(iii) $\circ^{-1} \xrightarrow{\zeta^7} \bullet \zeta^8$. The positive roots are the same as in (ii) exchanging 1 by 2, and $\mathcal{B}(V)$ admits an analogous presentation.

(iv) $\circ \zeta^9 \xrightarrow{\zeta} \bullet^{-1}$. The positive roots are

$$\Delta_+^\chi = \{\alpha_1, 3\alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, \alpha_1 + \alpha_2, \alpha_2\}.$$

Therefore $\mathcal{B}(V)$ has a presentation by generators x_1, x_2 , and relations

$$x_1^4 = x_2^2 = [x_{2\alpha_1+\alpha_2}, x_{12}]_c = 0.$$

(v) $\circ^{-1} \xrightarrow{\zeta^5} \bullet \zeta^9$. The set of positive roots is the same as in (iv) exchanging 1 by 2, and $\mathcal{B}(V)$ has an analogous presentation.

Example 2.4. Row 8. Let $\zeta \in \mathbb{G}'_{12}$ be primitive. The roots in $\mathcal{O}(\chi)$ are those such that $q_\alpha = \zeta^5$. For each one of these vector spaces V , $\dim \mathcal{B}(V) = 2^4 3^3 = 432$.

(i) $\circ \zeta^8 \xrightarrow{\zeta} \circ \zeta^8$. Their positive roots are

$$\Delta_+^\chi = \{\alpha_1, 2\alpha_1 + \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, \alpha_2\}.$$

According to the Theorem 1.6, $\mathcal{B}(V)$ has a presentation by generators x_1, x_2 , and relations

$$x_1^3 = x_2^3 = x_{\alpha_1+\alpha_2}^{12} = [x_1, x_{\alpha_1+2\alpha_2}]_c - \frac{\zeta^{10}(1-\zeta^7)q_{12}}{1-\zeta^9} x_{\alpha_1+\alpha_2}^2 = 0.$$

(ii) $\circ \zeta^8 \xrightarrow{\zeta^3} \circ^{-1}$. In this case,

$$\Delta_+^\chi = \{\alpha_1, 2\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2, \alpha_1 + \alpha_2, \alpha_2\}.$$

With the corresponding hyperwords from Remark 2.1, $\mathcal{B}(V)$ has a presentation by generators x_1, x_2 , and relations

$$x_1^3 = x_2^2 = x_{\alpha_1+\alpha_2}^{12} = [x_{3\alpha_1+2\alpha_2}, x_{12}]_c = 0.$$

(iii) $\circ \zeta^9 \xrightarrow{\zeta} \circ^{-1}$. The positive roots for this braiding are

$$\Delta_+^\chi = \{\alpha_1, 3\alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, \alpha_1 + \alpha_2, \alpha_2\}.$$

Therefore $\mathcal{B}(V)$ has a presentation by generators x_1, x_2 , and relations

$$x_1^{12} = x_2^2 = (\text{ad}_c x_1)^4 x_2 = [x_{2\alpha_1+\alpha_2}, x_{12}]_c = 0.$$

Example 2.5. Row 9. Let $\zeta \in \mathbb{G}'_9$ be primitive. The roots in $\mathcal{O}(\chi)$ are those such that $N_\alpha = 18$. For each one of these vector spaces V we have $\dim \mathcal{B}(V) = 2^4 3^6$.

(i) $\circ^{-\zeta} \xrightarrow{\zeta^7} \circ \zeta^3$. Their positive roots are

$$\Delta_+^\chi = \{\alpha_1, 2\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, \alpha_2\}.$$

Following Theorem 1.6, $\mathcal{B}(V)$ has a presentation by generators x_1, x_2 , and relations

$$x_1^{18} = x_2^3 = x_{\alpha_1+\alpha_2}^{18} = [x_1, x_{\alpha_1+2\alpha_2}]_c + \frac{\zeta^5(1-\zeta)q_{12}}{1-\zeta^7} x_{\alpha_1+\alpha_2}^2 = 0.$$

(ii) $\circ \zeta^3 \xrightarrow{\zeta^8} \circ^{-1}$. The set of positive roots is the following

$$\Delta_+^\chi = \{\alpha_1, 2\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2, 4\alpha_1 + 3\alpha_2, \alpha_1 + \alpha_2, \alpha_2\}.$$

Therefore $\mathcal{B}(V)$ has a presentation by generators x_1, x_2 , and relations

$$x_1^3 = x_2^2 = x_{\alpha_1 + \alpha_2}^{18} = x_{2\alpha_1 + \alpha_2}^{18} = [x_{2\alpha_1 + \alpha_2}, x_{3\alpha_1 + 2\alpha_2}]_c = 0.$$

(iii) ${}_{\circ} - \zeta^2 \xrightarrow{\zeta} {}_{\circ}^{-1}$. In this case,

$$\Delta_+^{\chi} = \{\alpha_1, 4\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, \alpha_1 + \alpha_2, \alpha_2\}.$$

Therefore $\mathcal{B}(V)$ has a presentation by generators x_1, x_2 , and relations

$$x_1^{18} = x_2^2 = x_{2\alpha_1 + \alpha_2}^{18} = (\text{ad}_c x_1)^5 x_2 = [x_{2\alpha_1 + \alpha_2}, x_{12}]_c = 0.$$

Example 2.6. Row 12. Let $\zeta \in \mathbb{G}'_{24}$ be primitive. Notice that the roots in $\mathcal{O}(\chi)$ are those such that $N_{\alpha} = 24$. For each one of these vector spaces V , we have $\dim \mathcal{B}(V) = 2^{10}3^4$.

(i) ${}_{\circ}\zeta^6 \xrightarrow{\zeta^{11}} {}_{\circ}\zeta^8$. The set of positive roots is

$$\{\alpha_1, 3\alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2, 4\alpha_1 + 3\alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, \alpha_2\}.$$

Considering the associated hyperwords, Theorem 1.6 establishes that $\mathcal{B}(V)$ has a presentation by generators x_1, x_2 , and relations

$$\begin{aligned} x_1^4 = x_2^3 = x_{3\alpha_1 + \alpha_2}^{24} = x_{\alpha_1 + \alpha_2}^{24} &= 0, \\ (1 - \zeta^{11})\zeta^4 q_{21} [x_1, x_{\alpha_1 + 2\alpha_2}]_c &= (1 - \zeta^{19})x_{\alpha_1 + \alpha_2}^2. \end{aligned}$$

(ii) ${}_{\circ}\zeta^6 \xrightarrow{\zeta} {}_{\circ}\zeta^{-1}$. The set of positive roots is in this case:

$$\{\alpha_1, 3\alpha_1 + \alpha_2, 5\alpha_1 + 2\alpha_2, 2\alpha_1 + \alpha_2, 5\alpha_1 + 3\alpha_2, 3\alpha_1 + 2\alpha_2, \alpha_1 + \alpha_2, \alpha_2\}.$$

Following Theorem 1.6, $\mathcal{B}(V)$ has a presentation by generators x_1, x_2 , and relations

$$x_1^4 = x_2^{24} = x_{2\alpha_1 + \alpha_2}^{24} = (\text{ad}_c x_2)^2 x_1 = [x_{3\alpha_1 + 2\alpha_2}, x_{12}]_c = 0.$$

(iii) ${}_{\circ}\zeta^8 \xrightarrow{\zeta^5} {}_{\circ}^{-1}$. The positive roots are in this case

$$\{\alpha_1, 2\alpha_1 + \alpha_2, 5\alpha_1 + 3\alpha_2, 3\alpha_1 + 2\alpha_2, 4\alpha_1 + 3\alpha_2, 5\alpha_1 + 4\alpha_2, \alpha_1 + \alpha_2, \alpha_2\}.$$

Therefore $\mathcal{B}(V)$ has a presentation by generators x_1, x_2 , and relations

$$\begin{aligned} x_1^3 = x_2^2 = x_{5\alpha_1 + 3\alpha_2}^{24} = x_{\alpha_1 + \alpha_2}^{24} &= 0, \\ [x_{2\alpha_1 + \alpha_2}, x_{4\alpha_1 + 3\alpha_2}]_c &= \frac{1 + \zeta + \zeta^6 + 2\zeta^7 + \zeta^{17}}{(1 + \zeta^4 + \zeta^6 + \zeta^{11})\zeta^{10} q_{21}} x_{3\alpha_1 + 2\alpha_2}^2. \end{aligned}$$

(iv) ${}_{\circ}\zeta \xrightarrow{\zeta^{19}} {}_{\circ}^{-1}$. The set of positive roots is in this case:

$$\{\alpha_1, 5\alpha_1 + \alpha_2, 4\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 5\alpha_1 + 2\alpha_2, 2\alpha_1 + \alpha_2, \alpha_1 + \alpha_2, \alpha_2\}.$$

Therefore $\mathcal{B}(V)$ has a presentation by generators x_1, x_2 , and relations

$$x_1^{24} = x_2^2 = x_{5\alpha_1 + 2\alpha_2}^{24} = (\text{ad}_c x_1)^6 x_2 = [x_{2\alpha_1 + \alpha_2}, x_{12}]_c = 0.$$

Example 2.7. Row 13. Let $\zeta \in \mathbb{G}'_5$ be primitive. The roots in $\mathcal{O}(\chi)$ are those such that $q_{\alpha} \in \{\zeta, -\zeta^3\}$. For each one of these vector spaces V , we have $\dim \mathcal{B}(V) = 2^6 5^4$.

(i) ${}_{\circ}\zeta \xrightarrow{\zeta^2} {}_{\circ}^{-1}$. The set of positive roots is in this case:

$$\{\alpha_1, 3\alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 5\alpha_1 + 3\alpha_2, 3\alpha_1 + 2\alpha_2, 4\alpha_1 + 3\alpha_2, \alpha_1 + \alpha_2, \alpha_2\}.$$

Following Theorem 1.6, $\mathcal{B}(V)$ has a presentation by generators x_1, x_2 , and relations

$$x_1^5 = x_2^2 = x_{2\alpha_1 + \alpha_2}^{10} = x_{3\alpha_1 + 2\alpha_2}^5 = x_{\alpha_1 + \alpha_2}^{10} = (\text{ad}_c x_1)^4 x_2 = [x_{4\alpha_1 + 3\alpha_2}, x_{12}]_c = 0.$$

(ii) ${}_o-\zeta^3 \xrightarrow{\zeta^3} {}_o^{-1}$. For this braiding, the positive roots are

$$\{\alpha_1, 4\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 5\alpha_1 + 2\alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2, \alpha_1 + \alpha_2, \alpha_2\}.$$

Therefore $\mathcal{B}(V)$ has a presentation by generators x_1, x_2 , and relations

$$\begin{aligned} x_1^{10} &= x_2^2 = x_{3\alpha_1+\alpha_2}^5 = x_{\alpha_1+\alpha_2}^5 = x_{2\alpha_1+\alpha_2}^{10} = 0, \\ (\text{ad}_c x_1)^5 x_2 &= [x_1, x_{3\alpha_1+2\alpha_2}]_c + q_{12} x_{2\alpha_1+\alpha_2}^2 = 0. \end{aligned}$$

Example 2.8. Row 14. Let $\zeta \in \mathbb{G}'_{20}$ be primitive. The roots in $\mathcal{O}(\chi)$ are those such that $N_\alpha = 20$. For each one of these vector spaces V , we have $\dim \mathcal{B}(V) = 2^8 5^4$.

(i) ${}_o\zeta \xrightarrow{\zeta^{17}} {}_o^{-1}$. The set of positive roots is the same as in Example 2.7,(i). Following Theorem 1.6, $\mathcal{B}(V)$ has a presentation by generators x_1, x_2 , and relations

$$x_1^{20} = x_2^2 = x_{3\alpha_1+2\alpha_2}^{20} = (\text{ad}_c x_1)^4 x_2 = [x_{4\alpha_1+3\alpha_2}, x_{12}]_c = 0.$$

(ii) ${}_o\zeta^{11} \xrightarrow{\zeta^7} {}_o^{-1}$. The set of positive roots is again the same as in Example 2.7,(i). According to Theorem 1.6, $\mathcal{B}(V)$ has a presentation by generators x_1, x_2 , and relations

$$x_1^{20} = x_2^2 = x_{3\alpha_1+2\alpha_2}^{20} = (\text{ad}_c x_1)^4 x_2 = [x_{4\alpha_1+3\alpha_2}, x_{12}]_c = 0.$$

(iii) ${}_o\zeta^8 \xrightarrow{\zeta^3} {}_o^{-1}$. The set of positive roots is the same as in Example 2.7,(ii). Following Theorem 1.6, $\mathcal{B}(V)$ has a presentation by generators x_1, x_2 , and relations

$$x_1^5 = x_2^2 = x_{3\alpha_1+\alpha_2}^{20} = x_{\alpha_1+\alpha_2}^{20} = [x_1, x_{3\alpha_1+2\alpha_2}]_c + \frac{(1-\zeta^{17})q_{12}}{1-\zeta^2} x_{2\alpha_1+\alpha_2}^2 = 0.$$

(iv) ${}_o\zeta^8 \xrightarrow{\zeta^{13}} {}_o^{-1}$. The set of positive roots is again the same as in Example 2.7,(ii). According to Theorem 1.6, $\mathcal{B}(V)$ has a presentation by generators x_1, x_2 , and relations

$$x_1^5 = x_2^2 = x_{3\alpha_1+\alpha_2}^{20} = x_{\alpha_1+\alpha_2}^{20} = [x_1, x_{3\alpha_1+2\alpha_2}]_c + \frac{(1-\zeta^7)q_{12}}{1-\zeta^2} x_{2\alpha_1+\alpha_2}^2 = 0.$$

Example 2.9. Row 15. Let $\zeta \in \mathbb{G}'_{15}$ be primitive. Notice that the roots in $\mathcal{O}(\chi)$ are those such that $N_\alpha = 30$. For each one of these vector spaces V , we have $\dim \mathcal{B}(V) = 2^4 3^4 5^4 = 30^4$.

(i) ${}_o-\zeta \xrightarrow{-\zeta^{12}} {}_o\zeta^5$. The set of positive roots is

$$\{\alpha_1, 3\alpha_1 + \alpha_2, 5\alpha_1 + 2\alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, \alpha_2\}.$$

Considering the associated hyperwords, Theorem 1.6 says that $\mathcal{B}(V)$ has a presentation by generators x_1, x_2 , and relations

$$\begin{aligned} x_1^{30} &= x_2^3 = x_{3\alpha_1+\alpha_2}^{30} = (\text{ad}_c x_1)^4 x_2 = 0, \\ [x_1, x_{\alpha_1+2\alpha_2}]_c &+ \frac{\zeta^{10}(1+\zeta^{13})q_{12}}{1+\zeta^{12}} x_{\alpha_1+\alpha_2}^2 = [x_{3\alpha_1+2\alpha_2}, x_{12}]_c = 0. \end{aligned}$$

(ii) ${}_o\zeta^3 \xrightarrow{-\zeta^4} {}_o-\zeta^{11}$. The set of positive roots is in this case:

$$\{\alpha_1, 4\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2, 4\alpha_1 + 3\alpha_2, \alpha_1 + \alpha_2, \alpha_2\}.$$

Following Theorem 1.6, $\mathcal{B}(V)$ has a presentation by generators x_1, x_2 , and relations

$$x_1^5 = x_2^{30} = x_{2\alpha_1+\alpha_2}^{30} = (\text{ad}_c x_2)^2 x_1 = 0,$$

$$[x_1, x_{3\alpha_1+2\alpha_2}]_c - \frac{(1-\zeta^2)\zeta^9 q_{12}}{1+\zeta^7} x_{2\alpha_1+\alpha_2}^2 = [x_{4\alpha_1+3\alpha_2}, x_{12}]_c = 0.$$

(iii) ${}_o\zeta^5 \xrightarrow{-\zeta^{13}} {}_o^{-1}$. The positive roots are in this case

$$\{\alpha_1, 2\alpha_1 + \alpha_2, 5\alpha_1 + 3\alpha_2, 8\alpha_1 + 5\alpha_2, 3\alpha_1 + 2\alpha_2, 4\alpha_1 + 3\alpha_2, \alpha_1 + \alpha_2, \alpha_2\}.$$

Therefore $\mathcal{B}(V)$ has a presentation by generators x_1, x_2 , and relations

$$x_1^3 = x_2^2 = x_{2\alpha_1+\alpha_2}^{30} = x_{4\alpha_1+3\alpha_2}^{30} = [x_{4\alpha_1+3\alpha_2}, x_{12}]_c = 0.$$

(iv) ${}_o\zeta^3 \xrightarrow{-\zeta^2} {}_o^{-1}$. The set of positive roots is in this case:

$$\{\alpha_1, 4\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 8\alpha_1 + 3\alpha_2, 5\alpha_1 + 2\alpha_2, 2\alpha_1 + \alpha_2, \alpha_1 + \alpha_2, \alpha_2\}.$$

Therefore $\mathcal{B}(V)$ has a presentation by generators x_1, x_2 , and relations

$$x_1^5 = x_2^2 = x_{4\alpha_1+\alpha_2}^{30} = x_{2\alpha_1+\alpha_2}^{30} = [x_{2\alpha_1+\alpha_2}, x_{12}]_c = [x_{5\alpha_1+2\alpha_2}, x_{112}]_c = 0.$$

Example 2.10. Row 16. Let $\zeta \in \mathbb{G}'_7$ be primitive. The roots in $\mathcal{O}(\chi)$ are those such that $q_\alpha \in \{-\zeta, -\zeta^5\}$. For each one of these vector spaces V , we have $\dim \mathcal{B}(V) = 2^{12}7^6$.

(i) ${}_o\zeta \xrightarrow{\zeta^2} {}_o^{-1}$. The set of positive roots is in this case:

$$\Delta_+^\chi = \{\alpha_1, 3\alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 7\alpha_1 + 4\alpha_2, 5\alpha_1 + 3\alpha_2, 8\alpha_1 + 5\alpha_2, \\ 3\alpha_1 + 2\alpha_2, 7\alpha_1 + 5\alpha_2, 4\alpha_1 + 3\alpha_2, 5\alpha_1 + 4\alpha_2, \alpha_1 + \alpha_2, \alpha_2\}.$$

Following Theorem 1.6, $\mathcal{B}(V)$ has a presentation by generators x_1, x_2 , and relations

$$x_2^2 = x_\alpha^{14} = 0, \quad \alpha = \alpha_1, 2\alpha_1 + \alpha_2, 5\alpha_1 + 3\alpha_2, 3\alpha_1 + 2\alpha_2, 4\alpha_1 + 3\alpha_2, \alpha_1 + \alpha_2,$$

$$(\text{ad}_c x_1)^4 x_2 = [x_{2\alpha_i+\alpha_j}, x_{4\alpha_i+3\alpha_j}]_c - \frac{(1-\zeta+3\zeta^4-\zeta^6)q_{ij}}{2\zeta+\zeta^2-\zeta^3-\zeta^4+\zeta^5-2} x_{3\alpha_i+2\alpha_j}^2 = 0,$$

(ii) ${}_o^{-\zeta^3} \xrightarrow{\zeta^3} {}_o^{-1}$. For this braiding, the positive roots are

$$\Delta_+^\chi = \{\alpha_1, 5\alpha_1 + \alpha_2, 4\alpha_1 + \alpha_2, 7\alpha_1 + 2\alpha_2, 3\alpha_1 + \alpha_2, 8\alpha_1 + 3\alpha_2, \\ 5\alpha_1 + 2\alpha_2, 7\alpha_1 + 3\alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2, \alpha_1 + \alpha_2, \alpha_2\}.$$

Therefore $\mathcal{B}(V)$ has a presentation by generators x_1, x_2 , and relations

$$x_2^2 = x_\alpha^{14} = 0, \quad \alpha = \alpha_1, 4\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 5\alpha_1 + 2\alpha_2, 2\alpha_1 + \alpha_2, \alpha_1 + \alpha_2,$$

$$(\text{ad}_c x_1)^6 x_2 = [x_1, x_{3\alpha_1+2\alpha_2}]_c + \frac{(1+\zeta^4)q_{12}}{1-\zeta^2} x_{2\alpha_1+\alpha_2}^2 = 0,$$

3. EXAMPLES IN RANK 3 AND 4

Now we consider three Weyl equivalences classes in rank three and two in rank four, and make the same work as in the previous Section.

Example 3.1. Rank 3, row 13. In this case we have two different Weyl groupoids, depending on the order of ζ , with the same root systems (the associated braidings are different). We analyze each case.

1. Let $\zeta \in \mathbb{G}'_3$ be primitive. The roots in $\mathcal{O}(\chi)$ are those such that $\text{ord } q_\alpha \in \{3, 6\}$. For each one of these vector spaces V , $\dim \mathcal{B}(V) = 2^4 3^6 6^3 = 2^7 3^9$.

(i) $\circ\zeta \xrightarrow{\zeta^2} \circ\zeta \xrightarrow{\zeta} \circ^{-1}$. The set of positive roots is the following:

$$\Delta_+^\chi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, 2\alpha_1 + 2\alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + 2\alpha_2 + \alpha_3, \\ \alpha_1 + 2\alpha_2 + 2\alpha_3, 2\alpha_2 + \alpha_3, \alpha_1 + 3\alpha_2 + 2\alpha_3, 2\alpha_1 + 3\alpha_2 + 2\alpha_3, 2\alpha_1 + 4\alpha_2 + 3\alpha_3\}.$$

According to Theorem 1.6, $\mathcal{B}(V)$ has a presentation by generators x_1, x_2, x_3 , and relations

$$x_3^2 = (\text{ad}_c x_1)^2 x_2 = (\text{ad}_c x_1) x_3 = (\text{ad}_c x_2)^2 x_1 = [x_{223}, x_{23}]_c = [[x_{123}, x_2]_c, x_2]_c = 0, \\ x_\alpha^3 = 0, \quad \alpha = \alpha_1, \alpha_1 + \alpha_2, \alpha_2, \alpha_1 + 2\alpha_2 + 2\alpha_3, \alpha_1 + 3\alpha_2 + 2\alpha_3, 2\alpha_1 + 3\alpha_2 + 2\alpha_3, \\ x_\alpha^6 = 0, \quad \alpha = \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + 2\alpha_2 + \alpha_3.$$

(ii) $\circ\zeta \xrightarrow{\zeta^2} \circ^{-\zeta^2} \xrightarrow{\zeta^2} \circ^{-1}$. In this case, we have

$$\Delta_+^\chi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, 2\alpha_1 + 2\alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3, 2\alpha_1 + 4\alpha_2 + \alpha_3, \\ 2\alpha_2 + \alpha_3, \alpha_2 + \alpha_3, \alpha_1 + 3\alpha_2 + \alpha_3, 2\alpha_1 + 3\alpha_2 + \alpha_3, \alpha_1 + 2\alpha_2 + \alpha_3\}.$$

Theorem 1.6 says that $\mathcal{B}(V)$ is presented by generators x_1, x_2, x_3 , and relations

$$x_3^2 = (\text{ad}_c x_1)^2 x_2 = (\text{ad}_c x_1) x_3 = (\text{ad}_c x_2)^3 x_1 = (\text{ad}_c x_2)^3 x_3 = 0, \\ [x_3, x_{221}]_c + q_{21}[x_{321}, x_2]_c + q_{32}\zeta^2(1 - \zeta^2)x_2x_{321} = 0, \\ x_\alpha^3 = 0, \quad \alpha = \alpha_1, \alpha_1 + \alpha_2 + \alpha_3, \alpha_2 + \alpha_3, \alpha_1 + 2\alpha_2, \alpha_1 + 3\alpha_2 + \alpha_3, 2\alpha_1 + 3\alpha_2 + \alpha_3, \\ x_\alpha^6 = 0, \quad \alpha = \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2 + \alpha_3.$$

2. Now, let $\zeta \in \mathbb{G}'_6$ be primitive. The roots in $\mathcal{O}(\chi)$ are those such that $\text{ord } q_\alpha = 6$. In this case, $\dim \mathcal{B}(V) = 2^4 3^3 6^6 = 2^{10} 3^9$.

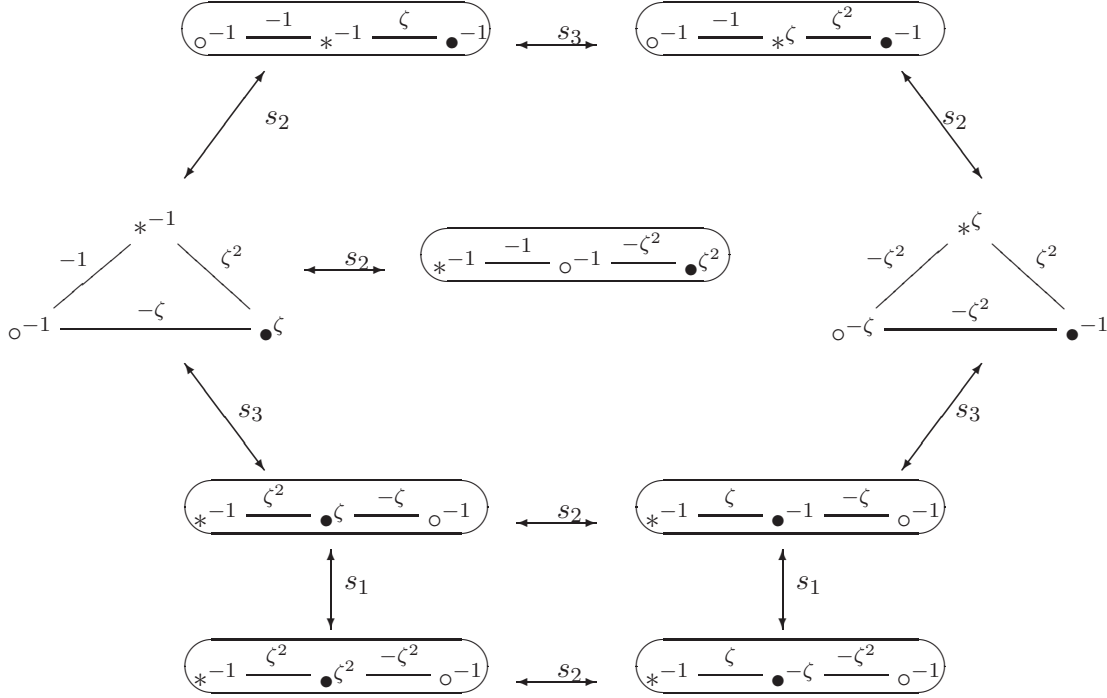
(i) $\circ\zeta \xrightarrow{\zeta^5} \circ\zeta \xrightarrow{\zeta^4} \circ^{-1}$. The set of positive roots is the same as in **1.(i)**. Using Theorem 1.6, we deduce that $\mathcal{B}(V)$ has a presentation by generators x_1, x_2, x_3 , and relations

$$x_3^2 = (\text{ad}_c x_1)^2 x_2 = (\text{ad}_c x_1) x_3 = (\text{ad}_c x_2)^2 x_1 = (\text{ad}_c x_2)^3 x_2 = 0, \\ x_\alpha^6 = 0, \quad \alpha = \alpha_1, \alpha_1 + \alpha_2, \alpha_2, \alpha_1 + 2\alpha_2 + 2\alpha_3, \alpha_1 + 3\alpha_2 + 2\alpha_3, 2\alpha_1 + 3\alpha_2 + 2\alpha_3.$$

(ii) $\circ\zeta \xrightarrow{\zeta^2} \circ^{-\zeta^2} \xrightarrow{\zeta^2} \circ^{-1}$. The positive roots are the same as in **1.(ii)**. By Theorem 1.6, $\mathcal{B}(V)$ is presented by generators x_1, x_2, x_3 , and relations

$$x_3^2 = x_2^3 = (\text{ad}_c x_1)^2 x_2 = (\text{ad}_c x_1) x_3 = [x_{223}, x_{23}]_c = 0, \\ [x_1, x_{223}]_c + q_{23}[x_{123}, x_2]_c - q_{12}x_2x_{123} = 0, \\ x_\alpha^6 = 0, \quad \alpha = \alpha_1, \alpha_1 + \alpha_2 + \alpha_3, \alpha_2 + \alpha_3, \alpha_1 + 2\alpha_2, \alpha_1 + 3\alpha_2 + \alpha_3, 2\alpha_1 + 3\alpha_2 + \alpha_3.$$

Example 3.2. Rank 3, row 17. Let $\zeta \in \mathbb{G}'_3$ be primitive. The following diagram shows the action of the Weyl groupoid, where we indicate the vertices 1, 2, 3 by \circ , $*$, \bullet , respectively, and we omit those symmetries which do not change the Dynkin diagram.



In this case, the roots in $\mathcal{O}(\chi)$ are those such that $N_\alpha = 6$, and $\dim \mathcal{B}(V) = 2^7 3^3 6 = 2^8 3^4$.

(i) $\circ^{-1} \xrightarrow{-1} *^{-1} \xrightarrow{\zeta} \bullet^{-1}$. We have that

$$\Delta_+^\chi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + 2\alpha_2 + \alpha_3, \alpha_1 + 2\alpha_2 + 2\alpha_3, \alpha_1 + 3\alpha_2 + 2\alpha_3, 2\alpha_1 + 3\alpha_2 + 2\alpha_3, 2\alpha_1 + 4\alpha_2 + 3\alpha_3\}.$$

Following Theorem 1.6, $\mathcal{B}(V)$ has a presentation by generators x_1, x_2, x_3 , and relations

$$x_1^2 = x_2^2 = x_3^2 = x_{12}^2 = (\text{ad}_c x_1)x_3 = [x_{32}, [x_{32}, x_{321}]_c]_c = x_{\alpha_1 + 2\alpha_2 + 2\alpha_3}^6 = 0.$$

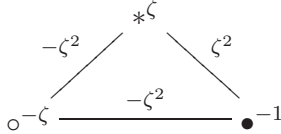
(ii) $\circ^{-1} \xrightarrow{-1} *\zeta \xrightarrow{\zeta^2} \bullet^{-1}$. For this braiding,

$$\Delta_+^\chi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + 3\alpha_2 + \alpha_3, 2\alpha_1 + 3\alpha_2 + \alpha_3, 2\alpha_1 + 4\alpha_2 + \alpha_3, \alpha_1 + 2\alpha_2 + \alpha_3\}.$$

By Theorem 1.6, $\mathcal{B}(V)$ has a presentation by generators x_1, x_2, x_3 , and relations

$$x_1^2 = x_2^3 = x_3^2 = (\text{ad}_c x_2)^2 x_3 = (\text{ad}_c x_1)x_3 = [x_{221}, x_{21}]_c = [x_{12}, x_{123}]_c = x_{\alpha_1 + 2\alpha_2}^6 = 0.$$

(iii) $\circ^{-\zeta} \xrightarrow{-\zeta^2} *\zeta \xrightarrow{\zeta^2} \bullet^{-1}$. The root system is



$$\Delta_+^\chi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + 2\alpha_2, \alpha_1 + \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, 2\alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + 2\alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3, 2\alpha_1 + 2\alpha_2 + \alpha_3\}.$$

By Theorem 1.6, $\mathcal{B}(V)$ admits a presentation by generators x_1, x_2, x_3 , and relations

$$\begin{aligned} x_1^6 &= x_2^3 = x_3^2 = (\text{ad}_c x_2)^2 x_3 = (\text{ad}_c x_1)^2 x_2 = (\text{ad}_c x_1)^2 x_3 = 0 \\ x_{123} &= -q_{23}(1 - \zeta^2)[x_{13}, x_2]_c + q_{12}(1 - \zeta^2)x_2 x_{13}. \end{aligned}$$

(iv) $*^{-1} \xrightarrow{\zeta} \bullet^{-1} \xrightarrow{-\zeta} \circ^{-1}$. In this case,

$$\begin{aligned} \Delta_+^\chi &= \{\alpha_1, \alpha_2, \alpha_3, \alpha_2 + \alpha_3, \alpha_3 + \alpha_1, \alpha_1 + 2\alpha_2 + 2\alpha_3, \alpha_1 + \alpha_2 + \alpha_3, 2\alpha_2 + 3\alpha_3 + \alpha_1, \\ &\quad \alpha_2 + 2\alpha_3 + \alpha_1, \alpha_2 + 2\alpha_3 + 2\alpha_1, 2\alpha_2 + 3\alpha_3 + 2\alpha_1\}. \end{aligned}$$

Following Theorem 1.6, $\mathcal{B}(V)$ has a presentation by generators x_1, x_2, x_3 , and relations

$$x_1^2 = x_2^2 = x_3^2 = (\text{ad}_c x_1)x_2 = [[x_{13}, x_{132}]_c, x_3]_c = x_{\alpha_1 + \alpha_3}^6 = 0.$$

(v) $*^{-1} \xrightarrow{\zeta} \bullet^{-\zeta} \xrightarrow{-\zeta^2} \circ^{-1}$. Its root system is

$$\begin{aligned} \Delta_+^\chi &= \{\alpha_1, \alpha_2, \alpha_3, \alpha_2 + \alpha_3, \alpha_2 + 2\alpha_3, \alpha_3 + \alpha_1, 2\alpha_2 + 2\alpha_3 + \alpha_1, \alpha_2 + \alpha_3 + \alpha_1, \\ &\quad 2\alpha_2 + 3\alpha_3 + \alpha_1, \alpha_2 + 2\alpha_3 + \alpha_1, 2\alpha_2 + 3\alpha_3 + 2\alpha_1\}. \end{aligned}$$

Following Theorem 1.6, $\mathcal{B}(V)$ has a presentation by generators x_1, x_2, x_3 , and relations

$$x_1^2 = x_2^2 = x_3^6 = (\text{ad}_c x_3)^3 x_2 = (\text{ad}_c x_3)^2 x_1 = (\text{ad}_c x_1)x_2 = 0.$$

(vi) $*^{-1} \xrightarrow{\zeta^2} \bullet^{\zeta^2} \xrightarrow{-\zeta^2} \circ^{-1}$. The corresponding root system is

$$\begin{aligned} \Delta_+^\chi &= \{\alpha_1, \alpha_2, \alpha_3, \alpha_2 + \alpha_3, \alpha_2 + 2\alpha_3, 2\alpha_3 + \alpha_1, \alpha_3 + \alpha_1, \alpha_2 + \alpha_3 + \alpha_1, \\ &\quad \alpha_2 + 3\alpha_3 + \alpha_1, \alpha_2 + 3\alpha_3 + 2\alpha_1, \alpha_2 + 2\alpha_3 + \alpha_1\}. \end{aligned}$$

By Theorem 1.6, it follows that $\mathcal{B}(V)$ is presented by generators x_1, x_2, x_3 , and relations

$$\begin{aligned} x_1^2 &= x_2^2 = x_3^3 = x_{\alpha_2 + \alpha_3}^6 = (\text{ad}_c x_1)x_2 = [x_{331}, x_{31}]_c = 0 \\ [x_1, x_{332}]_c &= -q_{23}^{-1}\zeta^2[x_{132}, x_3]_c + q_{13}x_3x_{132}. \end{aligned}$$

(vii) $*^{-1} \xrightarrow{\zeta^2} \bullet^\zeta \xrightarrow{-\zeta} \circ^{-1}$. We have the following positive roots:

$$\begin{aligned} \Delta_+^\chi &= \{\alpha_1, \alpha_2, \alpha_3, \alpha_2 + \alpha_3, 2\alpha_3 + \alpha_1, \alpha_3 + \alpha_1, \alpha_2 + \alpha_3 + \alpha_1, \alpha_2 + 3\alpha_3 + \alpha_1, \\ &\quad \alpha_2 + 2\alpha_3 + 2\alpha_1, \alpha_2 + 2\alpha_3 + \alpha_1, \alpha_2 + 3\alpha_3 + 2\alpha_1\}. \end{aligned}$$

Theorem 1.6 implies that $\mathcal{B}(V)$ has a presentation by generators x_1, x_2, x_3 , and relations

$$x_1^2 = x_2^2 = x_3^3 = x_{\alpha_2 + \alpha_3 + \alpha_1}^6 = (\text{ad}_c x_3)^2 x_2 = (\text{ad}_c x_1)x_2 = [x_{331}, x_{31}]_c = 0.$$

(viii) $\begin{array}{ccc} & *^{-1} & \\ -1 \swarrow & & \searrow \zeta^2 \\ \circ^{-1} & \xrightarrow{-\zeta} & \bullet^\zeta \end{array}$. For this braiding,

$$\begin{aligned} \Delta_+^\chi &= \{\alpha_1, \alpha_2, \alpha_3, \alpha_2 + \alpha_1, \alpha_2 + \alpha_3, \alpha_1 + \alpha_3, \alpha_1 + 2\alpha_3, \alpha_2 + \alpha_1 + \alpha_3, \\ &\quad \alpha_2 + \alpha_1 + 2\alpha_3, \alpha_2 + 2\alpha_1 + 2\alpha_3, \alpha_2 + 2\alpha_1 + 3\alpha_3\}. \end{aligned}$$

Following Theorem 1.6, $\mathcal{B}(V)$ has a presentation by generators x_1, x_2, x_3 , and relations

$$\begin{aligned} x_1^2 &= x_2^2 = x_3^3 = x_{12}^2 = x_{\alpha_2 + \alpha_1 + 2\alpha_3}^6 = (\text{ad}_c x_3)^3 x_2 = [x_{331}, x_{31}]_c = 0 \\ x_{123} &= q_{23}(\zeta - \zeta^2)[x_{13}, x_2]_c + q_{12}(1 - \zeta)x_2 x_{13}. \end{aligned}$$

(ix) $*^{-1} \xrightarrow{-1} \circ^{-1} \xrightarrow{-\zeta^2} \bullet \zeta^2$. Its positive roots are

$$\Delta_+^\chi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_2 + \alpha_1, \alpha_1 + \alpha_3, \alpha_1 + 2\alpha_3, \alpha_2 + \alpha_1 + \alpha_3, \alpha_2 + \alpha_1 + 2\alpha_3, \\ \alpha_2 + 2\alpha_1 + \alpha_3, \alpha_2 + 2\alpha_1 + 3\alpha_3, \alpha_2 + 2\alpha_1 + 2\alpha_3\}.$$

According to Theorem 1.6, $\mathcal{B}(V)$ is presented by generators x_1, x_2, x_3 , and relations

$$x_1^2 = x_2^2 = x_3^3 = x_{\alpha_2+2\alpha_1+2\alpha_3}^6 = (\text{ad}_c x_2)x_3 = [x_{331}, x_{31}]_c = [x_{3312}, x_{312}]_c = 0 \\ x_{12}^2 = [[x_{31}, x_{312}]_c, x_1]_c = 0.$$

Example 3.3. Rank 3, row 18. Let $\zeta \in \mathbb{G}'_9$ be primitive. We distinguish this Weyl equivalence class of root systems because all the other unidentified cases in rank 3 have elements of \mathbb{G}'_6 labelling their vertices and edges.

In this case, the roots in $\mathcal{O}(\chi)$ are those such that $N_\alpha = 9$, and $\dim \mathcal{B}(V) = 9^9 3^4 = 3^{22}$.

(i) $\circ \zeta \xrightarrow{\zeta^8} \circ \zeta \xrightarrow{\zeta^8} \circ \zeta^6$. We have that

$$\Delta_+^\chi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + 2\alpha_3, \alpha_1 + 2\alpha_2 + 3\alpha_3, \\ \alpha_2 + 2\alpha_3, \alpha_1 + 2\alpha_2 + 4\alpha_3, \alpha_1 + 3\alpha_2 + 4\alpha_3, \alpha_1 + 2\alpha_2 + 2\alpha_3, 2\alpha_1 + 3\alpha_2 + 4\alpha_3\}.$$

Following Theorem 1.6, $\mathcal{B}(V)$ has a presentation by generators x_1, x_2, x_3 , and relations

$$x_3^3 = (\text{ad}_c x_1)^2 x_2 = (\text{ad}_c x_1)x_3 = (\text{ad}_c x_2)^2 x_1 = (\text{ad}_c x_2)^2 x_3 = 0; \\ x_\alpha^9 = 0, \quad \alpha \neq \alpha_3, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + 2\alpha_2 + 3\alpha_3;$$

(ii) $\circ \zeta \xrightarrow{\zeta^8} \circ \zeta^5 \xrightarrow{\zeta^4} \circ \zeta^6$. For this braiding,

$$\Delta_+^\chi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + 2\alpha_2, \alpha_2 + 2\alpha_3, \alpha_1 + \alpha_2 + 2\alpha_3, \\ \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + 2\alpha_2 + \alpha_3, \alpha_1 + 3\alpha_2 + 2\alpha_3, \alpha_1 + 2\alpha_2 + 2\alpha_3, 2\alpha_1 + 3\alpha_2 + 2\alpha_3\}.$$

Therefore $\mathcal{B}(V)$ has a presentation by generators x_1, x_2, x_3 , and relations

$$x_3^3 = (\text{ad}_c x_1)^2 x_2 = (\text{ad}_c x_1)x_3 = (\text{ad}_c x_2)^3 x_1 = (\text{ad}_c x_2)^2 x_3 = 0; \\ x_\alpha^9 = 0, \quad \alpha \neq \alpha_3, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + 2\alpha_2 + \alpha_3;$$

Example 3.4. Rank 4, row 14. Let $q \in \mathbf{k}^\times$, $q \neq \pm 1$. The roots in $\mathcal{O}(\chi)$ are those such that $q_\alpha \in \{q, -q^{-1}\}$. Also, $\mathcal{B}(V)$ is finite-dimensional iff q has finite order. In such case, if $M = \text{ord } q$, $N = \text{ord } -q^{-1}$, we have $\dim \mathcal{B}(V) = 2^9 M^3 N^3$.

(i) $\circ^q \xrightarrow{q^{-1}} \circ^q \xrightarrow{q^{-1}} \circ^{-1} \xrightarrow{-q} \circ^{-q^{-1}}$. For this braiding,

$$\Delta_+^\chi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3, \alpha_2 + \alpha_3 + \alpha_4, \\ \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4, \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4, \\ \alpha_2 + 2\alpha_3 + \alpha_4, \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4, \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4\}.$$

Following Theorem 1.6, $\mathcal{B}(V)$ has a presentation by generators x_1, x_2, x_3, x_4 , and relations

$$(\text{ad}_c x_1)^2 x_2 = (\text{ad}_c x_2)^2 x_1 = (\text{ad}_c x_2)^2 x_3 = (\text{ad}_c x_4)^2 x_3 = 0; \\ x_3^2 = (\text{ad}_c x_1)x_3 = (\text{ad}_c x_1)x_4 = (\text{ad}_c x_2)x_4 = 0; \\ x_\alpha^M = 0, \quad \alpha = \alpha_1, \alpha_2, \alpha_1 + \alpha_2 + \alpha_3; \\ x_\alpha^N = 0, \quad \alpha = \alpha_4, \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4, \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4.$$

(ii) $\circ^q \xrightarrow{q^{-1}} \circ^{-1} \xrightarrow{q} \circ^{-1}$. In this case,

$$\begin{array}{c} \circ^{-1} \\ \begin{array}{c} \downarrow -1 \\ \swarrow -q^{-1} \end{array} \end{array}$$

$$\begin{aligned} \Delta_+^\chi = \{ & \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_3 + \alpha_4, \alpha_2 + \alpha_4, \alpha_2 + \alpha_3 + \alpha_4, \\ & \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_4, \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4, \\ & \alpha_1 + 2\alpha_2 + \alpha_4, \alpha_1 + 2\alpha_2 + \alpha_3 + 2\alpha_4, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \}. \end{aligned}$$

Therefore $\mathcal{B}(V)$ has a presentation by generators x_1, x_2, x_3, x_4 , and relations

$$\begin{aligned} (\text{ad}_c x_1)^2 x_2 &= (\text{ad}_c x_1) x_3 = (\text{ad}_c x_1) x_4 = [(\text{ad}_c x_1)(\text{ad}_c x_2) x_3, x_2]_c = 0; \\ x_2^2 &= x_3^2 = x_4^2 = ((\text{ad}_c x_2) x_4)^2 = 0; \\ [x_2, (\text{ad}_c x_3) x_4]_c &+ \frac{1}{2}(1+q)q_{43} [(\text{ad}_c x_2) x_4, x_3]_c - q_{23}(1+q^{-1})x_3(\text{ad}_c x_2) x_4 = 0; \\ x_\alpha^M &= 0, \quad \alpha = \alpha_1, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3; \\ x_\alpha^N &= 0, \quad \alpha = \alpha_3 + \alpha_4, \alpha_1 + 2\alpha_2 + \alpha_4, \alpha_1 + 2\alpha_2 + \alpha_3 + 2\alpha_4. \end{aligned}$$

(iii) $\circ^q \xrightarrow{q^{-1}} \circ^{-1} \xrightarrow{-1} \circ^{-1} \xrightarrow{-q} \circ^{-q^{-1}}$. For this braiding,

$$\begin{aligned} \Delta_+^\chi = \{ & \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3, \alpha_2 + \alpha_3 + \alpha_4, \\ & \alpha_1 + 2\alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4, \\ & \alpha_2 + 2\alpha_3 + \alpha_4, \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4 \}. \end{aligned}$$

Then $\mathcal{B}(V)$ has a presentation by generators x_1, x_2, x_3, x_4 , and relations

$$\begin{aligned} x_2^2 &= x_3^2 = ((\text{ad}_c x_2) x_3)^2 = (\text{ad}_c x_i)^2 x_j = 0, \quad i \neq 2, 3; \\ x_\alpha^M &= 0, \quad \alpha = \alpha_1, \alpha_2 + 2\alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4; \\ x_\alpha^N &= 0, \quad \alpha = \alpha_4, \alpha_1 + 2\alpha_2 + \alpha_3, \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4. \end{aligned}$$

Example 3.5. Rank 4, row 22. Let $\zeta \in \mathbb{G}'_4$. This Weyl equivalence class contains eight different diagrams:

$$\begin{aligned} \text{(i)} \quad & \circ^{-\zeta} \xrightarrow{\zeta} \circ^{-1} \xrightarrow{-\zeta} \circ^\zeta \xrightarrow{-\zeta} \circ^{-\zeta} & \text{(ii)} \quad & \circ^{-1} \xrightarrow{-\zeta} \circ^{-1} \xrightarrow{\zeta} \circ^{-1} \xrightarrow{\zeta} \circ^{-\zeta} \\ \text{(iii)} \quad & \circ^{-\zeta} \xrightarrow{\zeta} \circ^{-1} \xrightarrow{-\zeta} \circ^\zeta \\ & \begin{array}{c} -1 \downarrow \swarrow \\ \circ^{-1} \end{array} \quad \begin{array}{c} \searrow \\ -\zeta \end{array} & \text{(iv)} \quad & \circ^{-1} \xrightarrow{-\zeta} \circ^\zeta \xrightarrow{-\zeta} \circ^{-1} \\ & \begin{array}{c} -1 \downarrow \swarrow \\ \circ^{-1} \end{array} \quad \begin{array}{c} \searrow \\ -\zeta \end{array} & \text{(v)} \quad & \circ^{-1} \xrightarrow{\zeta} \circ^{-\zeta} \xrightarrow{\zeta} \circ^{-1} \xrightarrow{\zeta} \circ^{-\zeta} \\ & \begin{array}{c} -1 \downarrow \swarrow \\ \circ^{-1} \end{array} \quad \begin{array}{c} \searrow \\ -\zeta \end{array} & \text{(vi)} \quad & \circ^{-1} \xrightarrow{\zeta} \circ^{-1} \xrightarrow{-\zeta} \circ^{-1} \\ & \begin{array}{c} -1 \downarrow \swarrow \\ \circ^{-1} \end{array} \quad \begin{array}{c} \searrow \\ -\zeta \end{array} & \text{(vii)} \quad & \circ^{-1} \xrightarrow{-\zeta} \circ^\zeta \xrightarrow{-1} \circ^{-1} \xrightarrow{\zeta} \circ^{-\zeta} \\ & \begin{array}{c} -1 \downarrow \swarrow \\ \circ^{-1} \end{array} \quad \begin{array}{c} \searrow \\ -\zeta \end{array} & \text{(viii)} \quad & \circ^{-1} \xrightarrow{\zeta} \circ^{-1} \xrightarrow{-1} \circ^{-1} \xrightarrow{\zeta} \circ^{-\zeta} \end{aligned}$$

Each one of the associated Nichols algebras has dimension 2^{42} . They are presented by generators x_1, x_2, x_3, x_4 , and relations:

- if $N_\alpha = 4$: $x_\alpha^{N_\alpha}$;
- if $q_{ii} = -1$: x_i^2 ;

- if $q_{ii} = q_{jj} = \widetilde{q}_{ij} = -1$ (diagrams (iii), (vi), (viii)): x_{ij}^2 ;
- if $\widetilde{q}_{ij} = 1$: $(\text{ad}_c x_i)x_j$;
- if $\widetilde{q}_{ij} = q_{ii}^{-1} = \pm\zeta$: $(\text{ad}_c x_i)^2 x_j$;
- if $\widetilde{q}_{ij} = -1$, $q_{ii} = \zeta$ (diagrams (iv), (vii)): $(\text{ad}_c x_i)^3 x_j$;
- if $\widetilde{q}_{ij} = \widetilde{q}_{kj} = -\zeta$, $\widetilde{q}_{ik} = -1$ (diagrams (iii), (iv), (vi)):

$$x_{ijk} - \zeta q_{jk}[x_{ik}, x_j]_c - q_{ij}(1 + \zeta)x_j x_{ik};$$
- if $-q_{jj} = \widetilde{q}_{ij}\widetilde{q}_{kj} = \widetilde{q}_{ik} = 1$ (diagrams (i), (ii), (iii), (vi)): $[x_{ijk}, x_j]_c$;
- if $q_{jj} = \widetilde{q}_{ij} = -\widetilde{q}_{kj} = \zeta$, $\widetilde{q}_{ik} = 1$ (diagram (i)): $[[x_{ijk}, x_j]_c, x_j]_c$;
- if $q_{ii} = q_{jj} = -1$, $-\widetilde{q}_{ij} = \widetilde{q}_{kj} = \zeta$, $\widetilde{q}_{ik} = 1$ (diagram (ii)): $[[x_{ij}, [x_{ij}, x_{ijk}]_c]_c, x_j]_c$;

4. THE OTHER UNIDENTIFIED NICHOLS ALGEBRAS

In this Section we will consider the remaining braidings of unidentified type. We will consider four different subfamilies, according with the shape of their generalized Dynkin diagrams. The first subfamily that we will study is closely related with diagrams of type D_5 , E_6 , E_7 , but they contain non-Cartan vertices labeled with -1 and small orders on the q_{ii} (in fact, they are roots of unity of order 3 or 4).

Theorem 4.1. *Let (V, c) a braided vector space of diagonal type of dimension $\theta \in \{5, 6, 7\}$, whose generalized Dynkin diagram belongs to rows 11, 14, 17, 19 or 21 of [H2, Table 4]. Then $\mathcal{B}(V)$ is presented by generators x_1, \dots, x_θ and relations*

- if $N_\alpha \neq 2$ or $N_\alpha = 2$, $\alpha = \alpha_i$: $x_\alpha^{N_\alpha}$;
- if i, j are such that $q_{ii} = \widetilde{q}_{ij} = q_{jj} = -1$: x_{ij}^2 ;
- if i, j are such that $\widetilde{q}_{ij} = 1$ (respectively, $\widetilde{q}_{ij} = q_{ii}^{-1} \neq -1$):

$$(\text{ad}_c x_i)x_j \quad (\text{resp. } (\text{ad}_c x_i)^2 x_j;$$
- if i, j, k are such that $\widetilde{q}_{ik} = 1$, $q_{jj} = -1$, $\widetilde{q}_{ij}\widetilde{q}_{jk} = 1$: $[x_{ijk}, x_j]_c$;
- if i, j, k are such that $\widetilde{q}_{ik} = \widetilde{q}_{ij} = \widetilde{q}_{jk} =: \zeta \in \mathbb{G}'_3$:

$$x_{ijk} - q_{jk}\zeta^2[x_{ik}, x_j]_c - q_{ij}(1 - \zeta)x_j x_{ik};$$
- if i, j, k are such that $\widetilde{q}_{ik} = \widetilde{q}_{jk} =: \zeta \in \mathbb{G}'_4$, $\widetilde{q}_{ij} = -1$:

$$x_{ijk} + q_{jk}\zeta[x_{ik}, x_j]_c - q_{ij}(1 - \zeta)x_j x_{ik}.$$

Proof. First, note that $a_{ij} \in \{0, -1\}$ for all these braidings; that is, for each pair $i \neq j$, either $\widetilde{q}_{ij} = 1$, or else $\widetilde{q}_{ij} \neq -1$, $q_{ii} \in \{-1, \widetilde{q}_{ij}^{-1}\}$. This explains why we only consider quantum Serre relations associated to $a_{ij} = 0, 1$. We derive also that the Cartan vertices are those labeled with $q_{ii} \neq -1$, which explains why we need only the power root vectors associated to simple roots, or to other roots such that $N_\alpha \neq 2$.

Finally we look at the other needed relations. As the a_{ij} 's take only two values, we need few extra relations. \square

The second subfamily seems close to type super $D(n)$, but with certain 'degeneration' and small order on the labels of the vertices (roots of unity of order 2, 3, 5). With these diagrams and the corresponding to the previous family we cover all the cases in rank 5, 6, 7.

Theorem 4.2. *Let (V, c) a braided vector space of diagonal type of dimension $\theta \in \{3, 4, 5, 6\}$, whose generalized Dynkin diagram belongs to row 15 of [H2, Table 2], or row 18 of [H2, Table 3], or rows 12, 13, 15 or 18 of [H2, Table 4]. Then $\mathcal{B}(V)$ is presented by generators x_1, \dots, x_θ and relations*

- ⊙ if $N_\alpha \neq 2$: $x_\alpha^{N_\alpha}$;
- ⊙ if $q_{ii} = -1$: x_i^2 ;
- ⊙ if i, j are such that $\widetilde{q_{ij}} = 1$: $(\text{ad}_c x_i)x_j$;
- ⊙ if i, j are such that $\widetilde{q_{ij}} = q_{ii}^{-1} \neq -1$: $(\text{ad}_c x_i)^2 x_j$;
- ⊙ if i, j are such that $\widetilde{q_{ij}} = q_{ii}^{-2} \in \mathbb{G}'_5$: $(\text{ad}_c x_i)^3 x_j$;
- ⊙ if i, j, k are such that $\widetilde{q_{ik}} = 1$, $q_{jj} = -1$, $\widetilde{q_{ij}}\widetilde{q_{jk}} = 1$: $[x_{ijk}, x_j]_c$;
- ⊙ if i, j, k are such that $\widetilde{q_{ik}} = \widetilde{q_{ij}} = \widetilde{q_{jk}} =: \zeta \in \mathbb{G}'_3$:

$$x_{ijk} - q_{jk}\zeta^2[x_{ik}, x_j]_c - q_{ij}(1 - \zeta)x_jx_{ik};$$

- ⊙ if i, j, k are such that $\widetilde{q_{ij}} =: \zeta \in \mathbb{G}'_5$, $\widetilde{q_{ik}} = \widetilde{q_{jk}} = \zeta^2$:

$$x_{ijk} - q_{jk}\zeta^3[x_{ik}, x_j]_c - q_{ij}(1 - \zeta^2)x_jx_{ik}.$$

- ⊙ if i, j, k are such that $\widetilde{q_{ik}} = 1$, $q_{jj} = \widetilde{q_{ij}} = \widetilde{q_{jk}}^2 \in \mathbb{G}'_3$:

$$[[x_{ijk}, x_j]_c, x_j]_c;$$

- ⊙ if i, j, k are such that $\widetilde{q_{ik}} = 1$, $q_{ii} = q_{jj} = -1$, $\widetilde{q_{jk}} = \widetilde{q_{ik}}^{-2}$:

$$[[x_{ij}, x_{ijk}]_c, x_j]_c.$$

Proof. For these braidings, $a_{ij} \in \{0, -1, -2\}$. Moreover, the Cartan vertices are those such that $q_{ii} \neq -1$. In fact, when $q_{ii} \in \mathbb{G}'_3$ and there exists j such that $a_{ij} = -2$, then $q_{ij} = q_{ii}$, so i is a Cartan vertex (but we do not need the corresponding quantum Serre relation). Therefore we need only the power root vectors associated to a simple root, or to other roots such that $N_\alpha \neq 2$.

The remaining relations we need to generate the ideal expresses the similarity with the super $D(n)$ case. \square

The following subfamily keeps certain similarity with diagrams of type super $F(4)$ for small orders on the labels of the vertices (3, 6).

Theorem 4.3. *Let (V, c) a braided vector space of diagonal type of dimension $\theta = 4$, whose generalized Dynkin diagram belongs to rows 20 or 21 of [H2, Table 3]. Then $\mathcal{B}(V)$ is presented by generators x_1, x_2, x_3, x_4 and relations*

- ⊙ if $N_\alpha = 3, 6$: $x_\alpha^{N_\alpha}$;
- ⊙ if $q_{ii} = -1$: x_i^2 ;
- ⊙ if i, j are such that $\widetilde{q_{ij}} = 1$: $(\text{ad}_c x_i)x_j$;
- ⊙ if i, j are such that $\widetilde{q_{ij}} = q_{ii}^{-1} \neq -1$: $(\text{ad}_c x_i)^2 x_j$;
- ⊙ if i, j are such that $\widetilde{q_{ij}} = q_{ii}^{-2}$, $q_{ii} \in \mathbb{G}'_6$: $(\text{ad}_c x_i)^3 x_j$;
- ⊙ if i, j, k are such that $q_{jj} = -1$, $\widetilde{q_{ij}}\widetilde{q_{jk}} = 1$, $\widetilde{q_{ik}} = 1$: $[x_{ijk}, x_j]_c$;
- ⊙ if i, j, k are such that $\widetilde{q_{ik}} = 1$, $q_{jj} = \widetilde{q_{ij}} = \widetilde{q_{jk}}^2 \in \mathbb{G}'_3$:

$$[[x_{ijk}, x_j]_c, x_j]_c;$$

- ⊙ if i, j, k are such that $\widetilde{q_{ik}} = 1$, $q_{ii} = q_{jj} = -1$, $\widetilde{q_{jk}} = \widetilde{q_{ik}} \in \mathbb{G}'_3$:

$$[[x_{ij}, x_{ijk}]_c, x_j]_c;$$

- ⊙ if i, j, k are such that $\widetilde{q_{ik}} = \widetilde{q_{ij}} = \widetilde{q_{jk}} =: \zeta \in \mathbb{G}'_3$:

$$x_{ijk} - q_{jk}\zeta^2[x_{ik}, x_j]_c - q_{ij}(1 - \zeta)x_jx_{ik};$$

- ⊙ if i, j, k are such that $\widetilde{q_{ik}} = 1$, $-q_{jj} = \widetilde{q_{ij}} = \widetilde{q_{jk}} =: \zeta \in \mathbb{G}'_3$:

$$[x_i, x_{jjk}]_c + q_{jk}[x_{ijk}, x_j]_c + q_{ij}(\zeta - \zeta^2)x_jx_{ijk}.$$

Proof. It is analogous to the proof of Theorem 4.2. \square

Remark 4.4. If the diagram of (V, c) belongs to row 20 of [H2, Table 3], then

$$\dim \mathcal{B}(V) = 2^{13}3^{10}.$$

On the other hand, if it belongs to row 21 of [H2, Table 3], then

$$\dim \mathcal{B}(V) = 2^{20}3^{16}.$$

The last subfamily contains only two Weyl equivalence classes, where the labels of the vertices are roots of unity of order 2, 3, 6.

Theorem 4.5. *Let (V, c) a braided vector space of diagonal type of dimension $\theta = 3$ (resp. $\theta = 4$), whose generalized Dynkin diagram belongs to row 16 of [H2, Table 2], (resp. row 17 of [H2, Table 3]). Then $\mathcal{B}(V)$ is presented by generators x_1, \dots, x_θ and relations*

- ⊙ if $N_\alpha = 6$: x_α^6 ;
- ⊙ if $N_i = 2, 3$: $x_i^{N_i}$;
- ⊙ if i, j are such that $\widetilde{q_{ij}} = 1$: $(\text{ad}_c x_i)x_j$;
- ⊙ if i, j are such that $\widetilde{q_{ij}} = q_{ii}^{-1} \neq -1$: $(\text{ad}_c x_i)^2 x_j$;
- ⊙ if i, j are such that $\widetilde{q_{ij}} = q_{ii} = q_{jj} = -1$: x_{ij}^2 ;
- ⊙ if i, j are such that $q_{ii} \in \mathbb{G}'_3$, $q_{jj} = \widetilde{q_{ij}} = -1$: $[x_{ij}, x_{ij}]_c$;
- ⊙ if i, j, k are such that $\widetilde{q_{ij}} = -\widetilde{q_{ik}} = -\widetilde{q_{jk}} =: \zeta \in \mathbb{G}'_3$:

$$x_{ijk} + q_{jk}\zeta^2[x_{ik}, x_j]_c + q_{ij}\zeta^2 x_j x_{ik};$$

- ⊙ if i, j, k are such that $\widetilde{q_{ij}} = -1$, $\widetilde{q_{ik}}^{-1} = -\widetilde{q_{jk}} =: \zeta \in \mathbb{G}'_3$:

$$x_{ijk} + \frac{1}{3}q_{jk}\zeta[x_{ik}, x_j]_c + q_{ij}\zeta^2 x_j x_{ik}.$$

Proof. Now $a_{ij} \in \{0, -1, -2\}$, and the Cartan vertices are those such that $q_{ii}^2, q_{ii}^3 \neq 1$. The proof follows then as in the previous cases. \square

Remark 4.6. If the diagram of (V, c) belongs to [H2, Table 2, row 16], $\dim \mathcal{B}(V) = 2^7 3^6$.

If it belongs to [H2, Table 3, row 17], then $\dim \mathcal{B}(V) = 2^{19} 3^{15}$.

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NICHOLS ALGEBRAS OF UNIDENTIFIED DIAGONAL TYPE

IVÁN ANGIO

ABSTRACT. The Nichols algebras of diagonal type with finite root system are either of standard, super or (yet) unidentified type. A concrete description of the defining relations of all those Nichols algebras was given in [A3]. In the present paper we use this result to give an explicit presentation of all Nichols algebras of unidentified type.

INTRODUCTION

The classification of braidings of diagonal type whose Nichols algebra has a finite number of roots was given in [H2]. This problem is related with the classification of finite-dimensional pointed Hopf algebras over abelian groups. The list of Heckenberger can be split off in three families:

- standard braidings, introduced in [AA];
- braidings of super type, see [AAY];
- a finite list of braidings whose connected components have rank less than eight.

We call them *unidentified*.

We recall that the Nichols algebra of a braided vector space (V, c) is a quotient of its tensor algebra by a suitable ideal $I(V)$. A crucial question involving these Nichols algebras is to obtain a minimal set of relations generating $I(V)$. Such problem was solved in [A1] for the first family, with a formula for the dimension of each Nichols algebra of standard type. A presentation for the second family was given in [Y] for the generic case, and in [AAY] for the non-generic case (except by some considerations for small orders on the entries of the braiding matrix).

A complete answer can be found in [A3], where the main result gives a list of relations satisfied by the generators of the Nichols algebras, depending on the matrix entries, see Theorem 1.6 below. This paper depends strongly on [A2], the key point to obtain the desired presentation. It remains the problem to identify the relations needed for each one of the braidings. This is instrumental for several important questions concerning pointed Hopf algebras, among them the explicit determination of all liftings and their representation theory.

In this paper, we deal with unidentified braidings and give a complete list of relations generating the defining ideal for the Nichols algebra of each braiding of this kind. For the small ranks we give also the list of positive roots for each case and the dimension.

The organization of the paper is the following. The first Section includes some notions about PBW bases and root systems of Nichols algebras of diagonal type, as well as the presentation by generators and relations from [A3]. The second Section is devoted to give explicitly the presentation for each diagram. We begin with some unidentified families in rank 2, 3 or 4, and finally we split the remaining braidings in families by considering certain similarities on the associated generalized Cartan matrices.

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Notation. \mathbf{k} will denote an algebraically closed field of characteristic zero. For each $N > 1$, \mathbb{G}_N will denote the group of N -roots of unity in \mathbf{k} , and \mathbb{G}'_N the corresponding subset of primitive roots of order N .

1. PRELIMINARIES

We will recall all the preliminary results and fix the notation that we will use along this work. They are related with the theory of PBW bases for braided Hopf algebras of diagonal type [Kh], the Weyl groupoid of diagonal braided vector spaces [H1] and the presentation of Nichols algebras by generators and relations [A3].

1.1. PBW bases for Nichols algebras of diagonal type.

We begin with the definition of a Nichols algebra associated to a braided vector space. To this end, fix a Hopf algebra H with bijective antipode, and denote by ${}^H_H\mathcal{YD}$ the category of left Yetter-Drinfeld modules over H . Let $V \in {}^H_H\mathcal{YD}$. The tensor algebra $T(V)$ admits a braiding extending $c : V \otimes V \rightarrow V \otimes V$, and under such braiding it has a unique structure of graded braided Hopf algebra in ${}^H_H\mathcal{YD}$ such that $V \subseteq \mathcal{P}(V)$ (i.e. $\Delta(x) = x \otimes 1 + 1 \otimes x$).

Definition 1.1. [AS3] Let \mathfrak{S} be the family of all the homogeneous Hopf ideals of $I \subseteq T(V)$ such that

- I is generated by homogeneous elements of degree ≥ 2 ,
- I is a Yetter-Drinfeld submodule of $T(V)$.

The *Nichols algebra* $\mathcal{B}(V)$ associated to V is the quotient of $T(V)$ by the biggest ideal $I(V)$ of \mathfrak{S} .

The definition does not depend on the realization of a braided vector space (V, c) as a Yetter-Drinfeld module. In particular we will consider braidings of diagonal type; that is, there exists a basis $\{x_i\}_{i \in I}$ of V and a family of non-zero scalars $(q_{ij})_{i,j \in I}$ such that $c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i$. These braided vector spaces are related with vector spaces over group algebras of finite abelian groups.

Fix (V, c) a braided vector space of diagonal type. We will describe, following [Kh], a particular PBW basis for each graded braided Hopf algebra $\mathcal{B} = \bigoplus_{n \in \mathbb{N}} \mathcal{B}^n$ generated by $\mathcal{B}^1 \cong V$ as an algebra.

Assume that V is finite dimensional, and denote $\theta = \dim V$. Fix a basis $X = \{x_1, \dots, x_\theta\}$ of V as above. \mathbb{X} will denote the set of words with letters in X . We consider the lexicographical order on \mathbb{X} . We can identify $\mathbf{k}\mathbb{X}$ with $T(V)$.

$T(V)$ admits a unique \mathbb{Z}^I -graduation as a braided Hopf algebra such that $\deg x_i = \alpha_i$, where $(\alpha_i)_{i \in I}$ is the canonical basis of \mathbb{Z}^I . Assume that $\dim V = \theta < \infty$. Let $\chi : \mathbb{Z}^\theta \times \mathbb{Z}^\theta \rightarrow \mathbf{k}^\times$ be the bicharacter determined by the condition

$$(1) \quad \chi(\alpha_i, \alpha_j) = q_{ij}, \quad \text{for each pair } 1 \leq i, j \leq \theta.$$

Then, for each pair of \mathbb{Z}^θ -homogeneous elements $u, v \in \mathbb{X}$,

$$(2) \quad c(u \otimes v) = q_{u,v} v \otimes u, \quad q_{u,v} = \chi(\deg u, \deg v) \in \mathbf{k}^\times.$$

Note that [AS3, Prop. 2.10] implies that $I(V)$ is a \mathbb{Z}^θ -homogeneous ideal, and then $\mathcal{B}(V)$ is \mathbb{Z}^θ -graded, see also [L, Prop. 1.2.3].

Definition 1.2. $u \in \mathbb{X} - \{1\}$ is a *Lyndon word* if for every decomposition $u = vw$, $v, w \in \mathbb{X} - \{1\}$, it holds that $u < w$. We will denote the set of all Lyndon words by L .

We know that any word $u \in \mathbb{X}$ admits a unique decomposition as a non-increasing product of Lyndon words:

$$(3) \quad u = l_1 l_2 \dots l_r, \quad l_i \in L, l_r \leq \dots \leq l_1.$$

It is called the *Lyndon decomposition* of $u \in \mathbb{X}$, and each $l_i \in L$ in (3) is called a *Lyndon letter* of u .

For each $u \in L - X$, the *Shirshov decomposition* of u is the decomposition $u = u_1 u_2$, $u_1, u_2 \in L$, such that u_2 is the smallest end of u between all the possible decompositions satisfying these conditions (it is easily proved that each Lyndon word admits at least one of such decompositions).

For a general braided vector space, the *braided bracket* of $x, y \in T(V)$ is defined by

$$(4) \quad [x, y]_c := \text{multiplication} \circ (\text{id} - c)(x \otimes y).$$

Using the previous decompositions and the identification of \mathbb{X} with a basis of $T(V)$, we can define a \mathbf{k} -linear endomorphism $[-]_c$ of $T(V)$ as follows:

$$[u]_c := \begin{cases} u, & \text{if } u = 1 \text{ or } u \in X; \\ [[v]_c, [w]_c]_c, & \text{if } u \in L, \ell(u) > 1, u = vw \text{ is the Shirshov decomposition}; \\ [u_1]_c \dots [u_t]_c, & \text{if } u \in \mathbb{X} - L \text{ and its Lyndon decomposition is } u = u_1 \dots u_t. \end{cases}$$

We will obtain PBW bases using this automorphism.

Definition 1.3. The *hyperletter* corresponding to $l \in L$ is the element $[l]_c$. An *hyperword* is a word written in hyperletters; a *monotone hyperword* is an hyperword $W = [u_1]_c^{k_1} \dots [u_m]_c^{k_m}$ such that $u_1 > \dots > u_m$.

A different order on \mathbb{X} , considered in [U] and used implicitly in [Kh] is the *deg-lex order*, defined as follows. For each pair $u, v \in \mathbb{X}$, we say that $u \succ v$ if $\ell(u) < \ell(v)$, or $\ell(u) = \ell(v)$ and $u > v$ for the lexicographical order. Such order is total. The empty word 1 is the maximal element for \succ , and this order is invariant by left and right multiplication.

In what follows I will denote a Hopf ideal, and $R = T(V)/I$. Let $\pi : T(V) \rightarrow R$ be the canonical projection. We set:

$$G_I := \{u \in \mathbb{X} : u \notin \mathbf{k}\mathbb{X}_{\succ u} + I\}.$$

Therefore, if $u \in G_I$ and $u = vw$, then $v, w \in G_I$. In this way, each $u \in G_I$ is a non-increasing product of Lyndon words of G_I .

Consider the set $S_I := G_I \cap L$, and define $h_I : S_I \rightarrow \{2, 3, \dots\} \cup \{\infty\}$ by the condition:

$$(5) \quad h_I(u) := \min \{t \in \mathbb{N} : u^t \in \mathbf{k}\mathbb{X}_{\succ u^t} + I\}.$$

Following [Kh] we have the following results.

Theorem 1.4. *The set*

$$\{[u_1]_c^{k_1} \dots [u_m]_c^{k_m} : m \in \mathbb{N}_0, u_1 > \dots > u_m, u_i \in S_I, 0 < k_i < h_I(u_i)\}$$

is a PBW basis of $H = T(V)/I$. □

Corollary 1.5. (i) *A word u belongs to G_I if and only if the hyperletter $[u]_c$ is not a linear combination of greater hyperwords $[w]_c$, $w \succ u$, whose hyperletters are in S_I , modulo I .* □

(ii) *If $v \in S_I$ is such that $h_I(v) < \infty$, then $q_{v,v}$ is a root of unity. Moreover, if $\text{ord } q_{v,v} = h$, then $h_I(v) = h$, and $[v]^h$ is a linear combination of hyperwords $[w]_c$, $w \succ v^h$.* □

Let Δ_+^V be the set of degrees of a PBW basis of $\mathcal{B}(V)$, counted with their multiplicities [H1]. We can see that it does not depend on the PBW basis, [H1, AA]. We can attach a Cartan scheme \mathcal{C} , a Weyl groupoid \mathcal{W} and a root system \mathcal{R} in the sense of [CH, HY], see [HS, Thms. 6.2, 6.9]. To this end, define for each $1 \leq i \neq j \leq \theta$,

$$(6) \quad -a_{ij} := \min \{n \in \mathbb{N}_0 : (n+1)_{q_{ii}}(1 - q_{ii}^n \widetilde{q_{ij}}) = 0\},$$

and set $a_{ii} = 2$. The symmetry $s_i \in \text{Aut}(\mathbb{Z}^\theta)$ is defined by the condition $s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i$.

Set $\widetilde{q}_{rs} = \chi(s_i(\alpha_r), s_i(\alpha_s))$. Let V_i be another vector space of the same dimension, and attach to it the matrix $\widetilde{\mathbf{q}} = (\widetilde{q}_{rs})$. By [H1],

$$\Delta_+^{V_i} = s_i(\Delta_+^V \setminus \{\alpha_i\}) \cup \{\alpha_i\}.$$

Therefore last equation lets us to define the Weyl groupoid of V , whose root system is defined by the sets $\Delta^{V'}$, V' obtained after to apply some reflections to the matrix of V . Those braided vector spaces obtained after to apply the symmetries s_i define the *Weyl equivalence class* of V .

When the root system is finite, we can prove that each root is real, and in consequence it has multiplicity one, see [CH].

1.2. A presentation by generators and relations of Nichols algebras of diagonal type. Fix as above a finite-dimensional braided vector space (V, c) of diagonal type, with braiding matrix $(q_{ij})_{1 \leq i, j \leq \theta}$, $\theta = \dim V$, and a basis x_1, \dots, x_θ of V such that $c(x_i \otimes x_j) = q_{ij}x_j \otimes x_i$. Let $\widetilde{\chi}$ be the bicharacter associated to (q_{ij}) .

We denote $\widetilde{q}_{ij} = q_{ij}q_{ji}$. The *generalized Dynkin diagram* of a matrix $(q_{ij})_{1 \leq i, j \leq \theta}$ is a graph with θ vertices, labeled with the scalars q_{ii} , and an arrow between the vertices i and j if $\widetilde{q}_{ij} \neq 1$, labeled with this scalar. For example, given $q \in \mathbf{k}^\times$, the matrices

$$\begin{pmatrix} q^2 & q^{-2} & 1 \\ 1 & q^2 & q^{-2} \\ 1 & 1 & q \end{pmatrix}, \begin{pmatrix} q^2 & q^{-1} & q \\ q^{-1} & q^2 & q^{-1} \\ q^{-1} & q^{-1} & q \end{pmatrix}$$

have the diagram: $\circ_{q^2} \xrightarrow{q^{-2}} \circ_{q^2} \xrightarrow{q^{-2}} \circ_q$. In fact, two braided vector spaces of diagonal type are *twist equivalent* [AS3] if they have the same generalized Dynkin diagram.

We denote also

$$x_{i_1 i_2 \dots i_k} = (\text{ad}_c x_{i_1}) \cdots (\text{ad}_c x_{i_{k-1}}) x_{i_k}, \quad i_j \in \{1, \dots, \theta\}.$$

For each $m \in \mathbb{N}$, we define the elements $x_{(m+1)\alpha_i + m\alpha_j} \in \mathcal{U}(\chi)$ recursively:

- if $m = 1$, $x_{2\alpha_i + \alpha_j} := (\text{ad}_c x_i)^2 x_j = x_{iij}$,
- $x_{(m+2)\alpha_i + (m+1)\alpha_j} := [x_{(m+1)\alpha_i + m\alpha_j}, x_{ij}]_c$.

Call x_α , $\alpha \in \Delta_+^V$, the generator of the Kharchenko's PBW basis. We denote

$$N_\alpha := \text{ord } \chi(\alpha, \alpha), \quad \text{if } \chi(\alpha, \alpha) \text{ is a root of unity.}$$

We give now the main result of [A3].

Theorem 1.6. *Assume that the root system Δ^χ is finite. Then $\mathcal{B}(V)$ admits a presentation by generators x_1, \dots, x_θ and relations:*

- (7) $x_\alpha^{N_\alpha}, \quad \alpha \in \mathcal{O}(\chi);$
- (8) $(\text{ad}_c x_i)^{m_{ij}+1} x_j, \quad q_{ii}^{m_{ij}+1} \neq 1;$
- (9) $x_i^{N_i}, \quad i \text{ is not a Cartan vertex};$

⊙ if $i, j \in \{1, \dots, \theta\}$ are such that $q_{ii} = \widetilde{q}_{ij} = q_{jj} = -1$, and there exists $k \neq i, j$ such that $\widetilde{q}_{ik}^2 \neq 1$ or $\widetilde{q}_{jk}^2 \neq 1$,

$$(10) \quad x_{ij}^2;$$

⊙ if $i, j, k \in \{1, \dots, \theta\}$ are such that $q_{jj} = -1$, $\widetilde{q}_{ik} = \widetilde{q}_{ij}\widetilde{q}_{jk} = 1$,

$$(11) \quad [x_{ijk}, x_j]_c;$$

⊙ if $i, j \in \{1, \dots, \theta\}$ are such that $q_{jj} = -1$, $q_{ii}\widetilde{q}_{ij} \in \mathbb{G}_6$, and also $q_{ii} \in \mathbb{G}_3$ or $m_{ij} \geq 3$,

$$(12) \quad [x_{ij}, x_{ij}]_c;$$

⊙ if $i, j, k \in \{1, \dots, \theta\}$ are such that $q_{ii} = \pm \widetilde{q}_{ij} \in \mathbb{G}_3$, $\widetilde{q}_{ik} = 1$, and also $-q_{jj} = \widetilde{q}_{ij}\widetilde{q}_{jk} = 1$ or $q_{jj}^{-1} = \widetilde{q}_{ij} = \widetilde{q}_{jk} \neq -1$,

$$(13) \quad [x_{ijk}, x_{ij}]_c;$$

⊙ if $i, j, k \in \{1, \dots, \theta\}$ are such that $\widetilde{q}_{ik}, \widetilde{q}_{ij}, \widetilde{q}_{jk} \neq 1$,

$$(14) \quad x_{ijk} - \frac{1 - \widetilde{q}_{jk}}{q_{kj}(1 - \widetilde{q}_{ik})} [x_{ik}, x_j]_c - q_{ij}(1 - \widetilde{q}_{jk}) x_j x_{ik};$$

⊙ if $i, j, k \in \{1, \dots, \theta\}$ are such that one of the following situations

- ∘ $q_{ii} = q_{jj} = -1$, $\widetilde{q}_{ij}^2 = \widetilde{q}_{jk}^{-1}$, $\widetilde{q}_{ik} = 1$, or
- ∘ $\widetilde{q}_{ij} = q_{jj} = -1$, $q_{ii} = -\widetilde{q}_{jk}^2 \in \mathbb{G}_3$, $\widetilde{q}_{ik} = 1$, or
- ∘ $q_{kk} = \widetilde{q}_{jk} = q_{jj} = -1$, $q_{ii} = -\widetilde{q}_{ij} \in \mathbb{G}_3$, $\widetilde{q}_{ik} = 1$, or
- ∘ $q_{jj} = -1$, $\widetilde{q}_{ij} = q_{ii}^{-2}$, $\widetilde{q}_{jk} = -q_{ii}^{-3}$, $\widetilde{q}_{ik} = 1$, or
- ∘ $q_{ii} = q_{jj} = q_{kk} = -1$, $\pm \widetilde{q}_{ij} = \widetilde{q}_{jk} \in \mathbb{G}_3$, $\widetilde{q}_{ik} = 1$,

$$(15) \quad [[x_{ij}, x_{ijk}]_c, x_j]_c;$$

⊙ if $i, j, k \in \{1, \dots, \theta\}$ are such that $q_{ii} = q_{jj} = -1$, $\widetilde{q}_{ij}^3 = \widetilde{q}_{jk}^{-1}$, $\widetilde{q}_{ik} = 1$,

$$(16) \quad [[x_{ij}, [x_{ij}, x_{ijk}]_c], x_j]_c;$$

⊙ if $i, j, k \in \{1, \dots, \theta\}$ are such that $q_{jj} = \widetilde{q}_{ij}^2 = \widetilde{q}_{jk} \in \mathbb{G}_3$, $\widetilde{q}_{ik} = 1$,

$$(17) \quad [[x_{ijk}, x_j]_c, x_j]_c;$$

⊙ if $i, j, k \in \{1, \dots, \theta\}$ are such that $q_{kk} = q_{jj} = \widetilde{q}_{ij}^{-1} = \widetilde{q}_{jk}^{-1} \in \mathbb{G}_9$, $\widetilde{q}_{ik} = 1$, $q_{ii} = q_{kk}^6$

$$(18) \quad [x_{iij}, x_{iijk}]_c, x_{ij}]_c;$$

⊙ if $i, j, k \in \{1, \dots, \theta\}$ are such that $q_{ii} = \widetilde{q}_{ij}^{-1} \in \mathbb{G}_9$, $q_{jj} = \widetilde{q}_{jk}^{-1} = q_{ii}^5$, $\widetilde{q}_{ik} = 1$, $q_{kk} = q_{ii}^6$

$$(19) \quad [[x_{ijk}, x_j]_c, x_k]_c - (1 + \widetilde{q}_{jk})^{-1} q_{jk} [[x_{ijk}, x_k]_c, x_j]_c;$$

⊙ if $i, j, k \in \{1, \dots, \theta\}$ are such that $q_{jj} = \widetilde{q}_{ij}^3 = \widetilde{q}_{jk} \in \mathbb{G}_4$, $\widetilde{q}_{ik} = 1$,

$$(20) \quad [[x_{ijk}, x_j]_c, x_j]_c, x_j]_c;$$

⊙ if $i, j, k \in \{1, \dots, \theta\}$ are such that $q_{ii} = \widetilde{q}_{ij} = -1$, $q_{jj} = \widetilde{q}_{jk}^{-1} \neq -1$, $\widetilde{q}_{ik} = 1$,

$$(21) \quad [x_{ij}, x_{ijk}]_c;$$

⊙ if $i, j, k \in \{1, \dots, \theta\}$ are such that $q_{ii} = q_{kk} = -1$, $\widetilde{q}_{ik} = 1$, $\widetilde{q}_{ij} \in \mathbb{G}_3$, $q_{jj} = -\widetilde{q}_{jk} = \pm \widetilde{q}_{ij}$,

$$(22) \quad [x_i, x_{jjk}]_c - (1 + q_{jj}^2) q_{kj}^{-1} [x_{ijk}, x_j]_c - (1 + q_{jj}^2)(1 + q_{jj}) q_{ij} x_j x_{ijk};$$

⊙ if $i, j, k, l \in \{1, \dots, \theta\}$ are such that $q_{jj}\widetilde{q}_{ij} = q_{jj}\widetilde{q}_{jk} = 1$, $q_{kk} = -1$, $\widetilde{q}_{ik} = \widetilde{q}_{il} = \widetilde{q}_{jl} = 1$, $\widetilde{q}_{jk}^2 = \widetilde{q}_{ik}^{-1} = q_{il}$,

$$(23) \quad [[x_{ijkl}, x_k]_c, x_j]_c, x_k]_c;$$

⊙ if $i, j, k, l \in \{1, \dots, \theta\}$ are such that $\widetilde{q}_{jk} = \widetilde{q}_{ij} = q_{jj}^{-1} \in \mathbb{G}'_4 \cup \mathbb{G}'_6$, $q_{ii} = q_{kk} = -1$, $\widetilde{q}_{ik} = \widetilde{q}_{il} = \widetilde{q}_{jl} = 1$, $\widetilde{q}_{jk}^3 = \widetilde{q}_{lk}$,

$$(24) \quad [[x_{ijk}, [x_{ijkl}, x_k]_c]_c, x_{jk}]_c;$$

⊙ if $i, j, k, l \in \{1, \dots, \theta\}$ are such that $q_{ll} = \widetilde{q}_{lk}^{-1} = q_{kk} = \widetilde{q}_{jk}^{-1} = q^2$, $\widetilde{q}_{ij} = q_{ii}^{-1} = q^3$ for some $q \in \mathbf{k}^\times$, $q_{jj} = -1$, $\widetilde{q}_{ik} = \widetilde{q}_{il} = \widetilde{q}_{jl} = 1$,

$$(25) \quad [[[x_{ijk}, x_j]_c, [x_{ijkl}, x_j]_c]_c, x_{jk}]_c;$$

⊙ if $i, j, k, l \in \{1, \dots, \theta\}$ are such that one of the following situations hold

- ∘ $q_{kk} = -1$, $q_{ii} = \widetilde{q}_{ij}^{-1} = q_{jj}^2$, $\widetilde{q}_{kl} = q_{ll}^{-1} = q_{jj}^3$, $\widetilde{q}_{jk} = q_{jj}^{-1}$, $\widetilde{q}_{ik} = \widetilde{q}_{il} = \widetilde{q}_{jl} = 1$, or
- ∘ $q_{ii} = \widetilde{q}_{ij}^{-1} = -q_{ll}^{-1} = -\widetilde{q}_{kl}$, $q_{jj} = \widetilde{q}_{jk} = q_{kk} = -1$, $\widetilde{q}_{ik} = \widetilde{q}_{il} = \widetilde{q}_{jl} = 1$,

$$(26) \quad [x_{ijkl}, x_j]_c, x_k]_c - q_{jk}(\widetilde{q}_{ij}^{-1} - q_{jj}) [x_{ijkl}, x_k]_c, x_j]_c;$$

⊙ if $i, j, k \in \{1, \dots, \theta\}$ are such that $\widetilde{q}_{jk} = 1$, $q_{ii} = \widetilde{q}_{ij} = -\widetilde{q}_{ik} \in \mathbb{G}_3$,

$$(27) \quad [x_i, [x_{ij}, x_{ik}]_c]_c + q_{jk}q_{ik}q_{ji} [x_{iik}, x_{ij}]_c + q_{ij} x_{ij} x_{iik};$$

⊙ if $i, j, k \in \{1, \dots, \theta\}$ are such that $q_{jj} = q_{kk} = \widetilde{q}_{jk} = -1$, $q_{ii} = -\widetilde{q}_{ij} \in \mathbb{G}_3$, $\widetilde{q}_{ik} = 1$,

$$(28) \quad [x_{ijk}, x_{ijk}]_c;$$

⊙ if $i, j \in \{1, \dots, \theta\}$ are such that $-q_{ii}, -q_{jj}, q_{ii}\widetilde{q}_{ij}, q_{jj}\widetilde{q}_{ij} \neq 1$,

$$(29) \quad (1 - \widetilde{q}_{ij})q_{jj}q_{ji} [x_i, [x_{ij}, x_j]_c]_c - (1 + q_{jj})(1 - q_{jj}\widetilde{q}_{ij})x_{ij}^2;$$

⊙ if $i, j \in \{1, \dots, \theta\}$ are such that either $m_{ij} \in \{4, 5\}$, or else $q_{jj} = -1$, $m_{ji} = 3$, $q_{ii} \in \mathbb{G}'_4$,

$$(30) \quad [x_i, x_{3\alpha_i+2\alpha_j}]_c - \frac{1 - q_{ii}\widetilde{q}_{ij} - q_{ii}^2\widetilde{q}_{ij}^2q_{jj}}{(1 - q_{ii}\widetilde{q}_{ij})q_{ji}}x_{ij}^2;$$

⊙ if $i, j \in \{1, \dots, \theta\}$ are such that $4\alpha_i + 3\alpha_j \notin \Delta_+^\chi$, $q_{jj} = -1$ or $m_{ji} \geq 2$, and also $m_{ij} \geq 3$, or $m_{ij} = 2$, $q_{ii} \in \mathbb{G}_3$,

$$(31) \quad x_{4\alpha_i+3\alpha_j} = [x_{3\alpha_i+2\alpha_j}, x_{ij}]_c;$$

⊙ if $i, j \in \{1, \dots, \theta\}$ are such that $3\alpha_i + 2\alpha_j \in \Delta_+^\chi$, $5\alpha_i + 3\alpha_j \notin \Delta_+^\chi$, and $q_{ii}^3\widetilde{q}_{ij}, q_{ii}^4\widetilde{q}_{ij} \neq 1$,

$$(32) \quad [x_{iij}, x_{3\alpha_i+2\alpha_j}]_c;$$

⊙ if $i, j \in \{1, \dots, \theta\}$ are such that $4\alpha_i + 3\alpha_j \in \Delta_+^\chi$, $5\alpha_i + 4\alpha_j \notin \Delta_+^\chi$,

$$(33) \quad x_{5\alpha_i+4\alpha_j} = [x_{4\alpha_i+3\alpha_j}, x_{ij}]_c;$$

⊙ if $i, j \in \{1, \dots, \theta\}$ are such that $5\alpha_i + 2\alpha_j \in \Delta_+^\chi$, $7\alpha_i + 3\alpha_j \notin \Delta_+^\chi$,

$$(34) \quad [[x_{iij}, x_{iij}], x_{iij}]_c;$$

⊙ if $i, j \in \{1, \dots, \theta\}$ are such that $q_{jj} = -1$, $5\alpha_i + 4\alpha_j \in \Delta_+^\chi$,

$$(35) \quad [x_{iij}, x_{4\alpha_i+3\alpha_j}]_c - \frac{b - (1 + q_{ii})(1 - q_{ii}\zeta)(1 + \zeta + q_{ii}\zeta^2)q_{ii}^6\zeta^4}{a q_{ii}^3q_{ij}^2q_{ji}^3}x_{3\alpha_i+2\alpha_j}^2,$$

where $\zeta = \widetilde{q}_{ij}$, $a = (1 - \zeta)(1 - q_{ii}^4\zeta^3) - (1 - q_{ii}\zeta)(1 + q_{ii})q_{ii}\zeta$, $b = (1 - \zeta)(1 - q_{ii}^6\zeta^5) - a q_{ii}\zeta$. \square

2. UNIDENTIFIED NICHOLS ALGEBRAS OF RANK 2

In the following Sections we consider the different Weyl equivalence classes of braided vector spaces of unidentified type. We divide the work depending on the dimension of such spaces. We consider in this Section the 2-dimensional unidentified spaces, then some particular cases in rank three and four, and finally the remaining cases, but dividing the work in some families, according with the shape of the associated generalized Dynkin diagram.

The unidentified braided vector spaces in [H2, Table 1] of rank two are those in rows 7, 8, 9, 12, 13, 14, 15 and 16.

We will consider each possible row, describe the root system of each braiding, and calculate the dimension of the corresponding Nichols algebra.

Remark 2.1. The hyperword associated to a simple root α_i is the one associated to the unique Lyndon word of degree α_i : x_i . Also, the hyperword associated to a root of the way $m\alpha_1 + \alpha_2$ is $x_{m\alpha_1 + \alpha_2} = (\text{ad}_c x_1)^m x_2$. Moreover, for the braidings considered in this Subsection we have the following possible hyperwords:

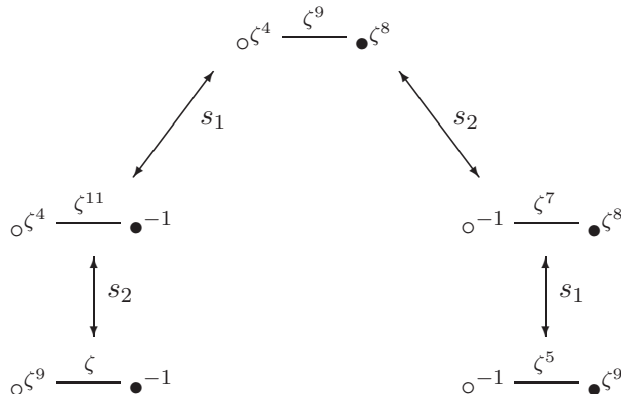
$$\begin{aligned} x_{\alpha_1 + 2\alpha_2} &= [x_{\alpha_1 + \alpha_2}, x_2]_c, & x_{3\alpha_1 + 2\alpha_2} &= [x_{2\alpha_1 + \alpha_2}, x_{\alpha_1 + \alpha_2}]_c, \\ x_{4\alpha_1 + 3\alpha_2} &= [x_{3\alpha_1 + 2\alpha_2}, x_{\alpha_1 + \alpha_2}]_c, & x_{5\alpha_1 + 2\alpha_2} &= [x_{3\alpha_1 + \alpha_2}, x_{2\alpha_1 + \alpha_2}]_c, \\ x_{5\alpha_1 + 3\alpha_2} &= [x_{2\alpha_1 + \alpha_2}, x_{3\alpha_1 + 2\alpha_2}]_c, & x_{5\alpha_1 + 4\alpha_2} &= [x_{4\alpha_1 + 3\alpha_2}, x_{\alpha_1 + \alpha_2}]_c, \\ x_{7\alpha_1 + 2\alpha_2} &= [x_{4\alpha_1 + \alpha_2}, x_{3\alpha_1 + \alpha_2}]_c, & x_{7\alpha_1 + 3\alpha_2} &= [x_{5\alpha_1 + 2\alpha_2}, x_{2\alpha_1 + \alpha_2}]_c, \\ x_{7\alpha_1 + 4\alpha_2} &= [x_{2\alpha_1 + \alpha_2}, x_{5\alpha_1 + 3\alpha_2}]_c, & x_{8\alpha_1 + 3\alpha_2} &= [x_{3\alpha_1 + \alpha_2}, x_{5\alpha_1 + 2\alpha_2}]_c, \\ x_{8\alpha_1 + 5\alpha_2} &= [x_{5\alpha_1 + 3\alpha_2}, x_{3\alpha_1 + 2\alpha_2}]_c. \end{aligned}$$

Remark 2.2. If V, W are two Weyl equivalent braided vector spaces of diagonal type, then $\dim \mathcal{B}(V) = \dim \mathcal{B}(W)$. It follows from the fact

$$\Delta_+^{s_p^* \chi} = s_p(\Delta^\chi \setminus \{\alpha_p\}) \cup \{\alpha_p\}$$

and that a hyperword of degree α has height $\text{ord } \chi(\alpha, \alpha)$, so we calculate the dimension computing the number of terms of the PBW basis; i.e. multiplying the orders of the associated scalars.

Example 2.3. Row 7. These are the first unidentified braided vector spaces, for which $\zeta \in \mathbb{G}'_{12}$ is primitive. The following diagram shows the action of the Weyl groupoid, where we indicate the vertices 1, 2 by \circ, \bullet , respectively. We omit those symmetries not changing the Dynkin diagram.



In this case, $\mathcal{O}(\chi)$ is empty. For each one of these vector spaces V , $\dim \mathcal{B}(V) = 2^4 3^2 = 144$.

(i) ${}_o\zeta^4 \xrightarrow{\zeta^9} \bullet \zeta^8$. In this case,

$$\Delta_+^\chi = \{\alpha_1, 2\alpha_1 + \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, \alpha_2\}.$$

Considering the corresponding hyperwords from Remark 2.1 and following Theorem 1.6, $\mathcal{B}(V)$ has a presentation by generators x_1, x_2 , and relations

$$x_1^3 = x_2^3 = [x_1, x_{\alpha_1+2\alpha_2}]_c - \frac{\zeta^{10}(1-\zeta^7)q_{12}}{1-\zeta^9} x_{\alpha_1+\alpha_2}^2 = 0.$$

(ii) ${}_o\zeta^4 \xrightarrow{\zeta^{11}} \bullet^{-1}$. The set of positive roots is in this case

$$\Delta_+^\chi = \{\alpha_1, 2\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2, \alpha_1 + \alpha_2, \alpha_2\}.$$

Therefore $\mathcal{B}(V)$ has a presentation by generators x_1, x_2 , and relations

$$x_1^3 = x_2^2 = [x_{3\alpha_1+2\alpha_2}, x_{12}]_c = 0.$$

(iii) ${}_o^{-1} \xrightarrow{\zeta^7} \bullet \zeta^8$. The positive roots are the same as in (ii) exchanging 1 by 2, and $\mathcal{B}(V)$ admits an analogous presentation.

(iv) ${}_o\zeta^9 \xrightarrow{\zeta} \bullet^{-1}$. The positive roots are

$$\Delta_+^\chi = \{\alpha_1, 3\alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, \alpha_1 + \alpha_2, \alpha_2\}.$$

Therefore $\mathcal{B}(V)$ has a presentation by generators x_1, x_2 , and relations

$$x_1^4 = x_2^2 = [x_{2\alpha_1+\alpha_2}, x_{12}]_c = 0.$$

(v) ${}_o^{-1} \xrightarrow{\zeta^5} \bullet \zeta^9$. The set of positive roots is the same as in (iv) exchanging 1 by 2, and $\mathcal{B}(V)$ has an analogous presentation.

Example 2.4. Row 8. Let $\zeta \in \mathbb{G}'_{12}$ be primitive. The roots in $\mathcal{O}(\chi)$ are those such that $q_\alpha = \zeta^5$. For each one of these vector spaces V , $\dim \mathcal{B}(V) = 2^4 3^3 = 432$.

(i) ${}_o\zeta^8 \xrightarrow{\zeta} {}_o\zeta^8$. Their positive roots are

$$\Delta_+^\chi = \{\alpha_1, 2\alpha_1 + \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, \alpha_2\}.$$

According to the Theorem 1.6, $\mathcal{B}(V)$ has a presentation by generators x_1, x_2 , and relations

$$x_1^3 = x_2^3 = x_{\alpha_1+\alpha_2}^{12} = [x_1, x_{\alpha_1+2\alpha_2}]_c - \frac{\zeta^{10}(1-\zeta^7)q_{12}}{1-\zeta^9} x_{\alpha_1+\alpha_2}^2 = 0.$$

(ii) ${}_o\zeta^8 \xrightarrow{\zeta^3} {}_o^{-1}$. In this case,

$$\Delta_+^\chi = \{\alpha_1, 2\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2, \alpha_1 + \alpha_2, \alpha_2\}.$$

With the corresponding hyperwords from Remark 2.1, $\mathcal{B}(V)$ has a presentation by generators x_1, x_2 , and relations

$$x_1^3 = x_2^2 = x_{\alpha_1+\alpha_2}^{12} = [x_{3\alpha_1+2\alpha_2}, x_{12}]_c = 0.$$

(iii) ${}_o\zeta^9 \xrightarrow{\zeta} {}_o^{-1}$. The positive roots for this braiding are

$$\Delta_+^\chi = \{\alpha_1, 3\alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, \alpha_1 + \alpha_2, \alpha_2\}.$$

Therefore $\mathcal{B}(V)$ has a presentation by generators x_1, x_2 , and relations

$$x_1^{12} = x_2^2 = (\text{ad}_c x_1)^4 x_2 = [x_{2\alpha_1+\alpha_2}, x_{12}]_c = 0.$$

Example 2.5. Row 9. Let $\zeta \in \mathbb{G}'_9$ be primitive. The roots in $\mathcal{O}(\chi)$ are those such that $N_\alpha = 18$. For each one of these vector spaces V we have $\dim \mathcal{B}(V) = 2^4 3^6$.

(i) ${}_{\circ-\zeta} \xrightarrow{\zeta^7} {}_{\circ}\zeta^3$. Their positive roots are

$$\Delta_+^\chi = \{\alpha_1, 2\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, \alpha_2\}.$$

Following Theorem 1.6, $\mathcal{B}(V)$ has a presentation by generators x_1, x_2 , and relations

$$x_1^{18} = x_2^3 = x_{\alpha_1+\alpha_2}^{18} = [x_1, x_{\alpha_1+2\alpha_2}]_c + \frac{\zeta^5(1-\zeta)q_{12}}{1-\zeta^7} x_{\alpha_1+\alpha_2}^2 = 0.$$

(ii) ${}_{\circ}\zeta^3 \xrightarrow{\zeta^8} {}_{\circ-1}$. The set of positive roots is the following

$$\Delta_+^\chi = \{\alpha_1, 2\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2, 4\alpha_1 + 3\alpha_2, \alpha_1 + \alpha_2, \alpha_2\}.$$

Therefore $\mathcal{B}(V)$ has a presentation by generators x_1, x_2 , and relations

$$x_1^3 = x_2^2 = x_{\alpha_1+\alpha_2}^{18} = x_{2\alpha_1+\alpha_2}^{18} = [x_{2\alpha_1+\alpha_2}, x_{3\alpha_1+2\alpha_2}]_c = 0.$$

(iii) ${}_{\circ-\zeta^2} \xrightarrow{\zeta} {}_{\circ-1}$. In this case,

$$\Delta_+^\chi = \{\alpha_1, 4\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, \alpha_1 + \alpha_2, \alpha_2\}.$$

Therefore $\mathcal{B}(V)$ has a presentation by generators x_1, x_2 , and relations

$$x_1^{18} = x_2^2 = x_{2\alpha_1+\alpha_2}^{18} = (\text{ad}_c x_1)^5 x_2 = [x_{2\alpha_1+\alpha_2}, x_{12}]_c = 0.$$

Example 2.6. Row 12. Let $\zeta \in \mathbb{G}'_{24}$ be primitive. Notice that the roots in $\mathcal{O}(\chi)$ are those such that $N_\alpha = 24$. For each one of these vector spaces V , we have $\dim \mathcal{B}(V) = 2^{10} 3^4$.

(i) ${}_{\circ}\zeta^6 \xrightarrow{\zeta^{11}} {}_{\circ}\zeta^8$. The set of positive roots is

$$\{\alpha_1, 3\alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2, 4\alpha_1 + 3\alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, \alpha_2\}.$$

Considering the associated hyperwords, Theorem 1.6 establishes that $\mathcal{B}(V)$ has a presentation by generators x_1, x_2 , and relations

$$\begin{aligned} x_1^4 = x_2^3 = x_{3\alpha_1+\alpha_2}^{24} = x_{\alpha_1+\alpha_2}^{24} &= 0, \\ (1 - \zeta^{11})\zeta^4 q_{21} [x_1, x_{\alpha_1+2\alpha_2}]_c &= (1 - \zeta^{19})x_{\alpha_1+\alpha_2}^2. \end{aligned}$$

(ii) ${}_{\circ}\zeta^6 \xrightarrow{\zeta} {}_{\circ}\zeta^{-1}$. The set of positive roots is in this case:

$$\{\alpha_1, 3\alpha_1 + \alpha_2, 5\alpha_1 + 2\alpha_2, 2\alpha_1 + \alpha_2, 5\alpha_1 + 3\alpha_2, 3\alpha_1 + 2\alpha_2, \alpha_1 + \alpha_2, \alpha_2\}.$$

Following Theorem 1.6, $\mathcal{B}(V)$ has a presentation by generators x_1, x_2 , and relations

$$x_1^4 = x_2^{24} = x_{2\alpha_1+\alpha_2}^{24} = (\text{ad}_c x_2)^2 x_1 = [x_{3\alpha_1+2\alpha_2}, x_{12}]_c = 0.$$

(iii) ${}_{\circ}\zeta^8 \xrightarrow{\zeta^5} {}_{\circ-1}$. The positive roots are in this case

$$\{\alpha_1, 2\alpha_1 + \alpha_2, 5\alpha_1 + 3\alpha_2, 3\alpha_1 + 2\alpha_2, 4\alpha_1 + 3\alpha_2, 5\alpha_1 + 4\alpha_2, \alpha_1 + \alpha_2, \alpha_2\}.$$

Therefore $\mathcal{B}(V)$ has a presentation by generators x_1, x_2 , and relations

$$\begin{aligned} x_1^3 = x_2^2 = x_{5\alpha_1+3\alpha_2}^{24} = x_{\alpha_1+\alpha_2}^{24} &= 0, \\ [x_{2\alpha_1+\alpha_2}, x_{4\alpha_1+3\alpha_2}]_c &= \frac{1 + \zeta + \zeta^6 + 2\zeta^7 + \zeta^{17}}{(1 + \zeta^4 + \zeta^6 + \zeta^{11})\zeta^{10}q_{21}} x_{3\alpha_1+2\alpha_2}^2. \end{aligned}$$

(iv) ${}_o\zeta \xrightarrow{\zeta^{19}} {}_o^{-1}$. The set of positive roots is in this case:

$$\{\alpha_1, 5\alpha_1 + \alpha_2, 4\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 5\alpha_1 + 2\alpha_2, 2\alpha_1 + \alpha_2, \alpha_1 + \alpha_2, \alpha_2\}.$$

Therefore $\mathcal{B}(V)$ has a presentation by generators x_1, x_2 , and relations

$$x_1^{24} = x_2^2 = x_{5\alpha_1+2\alpha_2}^{24} = (\text{ad}_c x_1)^6 x_2 = [x_{2\alpha_1+\alpha_2}, x_{12}]_c = 0.$$

Example 2.7. Row 13. Let $\zeta \in \mathbb{G}'_5$ be primitive. The roots in $\mathcal{O}(\chi)$ are those such that $q_\alpha \in \{\zeta, -\zeta^3\}$. For each one of these vector spaces V , we have $\dim \mathcal{B}(V) = 2^6 5^4$.

(i) ${}_o\zeta \xrightarrow{\zeta^2} {}_o^{-1}$. The set of positive roots is in this case:

$$\{\alpha_1, 3\alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 5\alpha_1 + 3\alpha_2, 3\alpha_1 + 2\alpha_2, 4\alpha_1 + 3\alpha_2, \alpha_1 + \alpha_2, \alpha_2\}.$$

Following Theorem 1.6, $\mathcal{B}(V)$ has a presentation by generators x_1, x_2 , and relations

$$x_1^5 = x_2^2 = x_{2\alpha_1+\alpha_2}^{10} = x_{3\alpha_1+2\alpha_2}^5 = x_{\alpha_1+\alpha_2}^{10} = (\text{ad}_c x_1)^4 x_2 = [x_{4\alpha_1+3\alpha_2}, x_{12}]_c = 0.$$

(ii) ${}_o^{-\zeta^3} \xrightarrow{\zeta^3} {}_o^{-1}$. For this braiding, the positive roots are

$$\{\alpha_1, 4\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 5\alpha_1 + 2\alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2, \alpha_1 + \alpha_2, \alpha_2\}.$$

Therefore $\mathcal{B}(V)$ has a presentation by generators x_1, x_2 , and relations

$$x_1^{10} = x_2^2 = x_{3\alpha_1+\alpha_2}^5 = x_{\alpha_1+\alpha_2}^5 = x_{2\alpha_1+\alpha_2}^{10} = 0, \\ (\text{ad}_c x_1)^5 x_2 = [x_1, x_{3\alpha_1+2\alpha_2}]_c + q_{12} x_{2\alpha_1+\alpha_2}^2 = 0.$$

Example 2.8. Row 14. Let $\zeta \in \mathbb{G}'_{20}$ be primitive. The roots in $\mathcal{O}(\chi)$ are those such that $N_\alpha = 20$. For each one of these vector spaces V , we have $\dim \mathcal{B}(V) = 2^8 5^4$.

(i) ${}_o\zeta \xrightarrow{\zeta^{17}} {}_o^{-1}$. The set of positive roots is the same as in Example 2.7,(i). Following Theorem 1.6, $\mathcal{B}(V)$ has a presentation by generators x_1, x_2 , and relations

$$x_1^{20} = x_2^2 = x_{3\alpha_1+2\alpha_2}^{20} = (\text{ad}_c x_1)^4 x_2 = [x_{4\alpha_1+3\alpha_2}, x_{12}]_c = 0.$$

(ii) ${}_o\zeta^{11} \xrightarrow{\zeta^7} {}_o^{-1}$. The set of positive roots is again the same as in Example 2.7,(i). According to Theorem 1.6, $\mathcal{B}(V)$ has a presentation by generators x_1, x_2 , and relations

$$x_1^{20} = x_2^2 = x_{3\alpha_1+2\alpha_2}^{20} = (\text{ad}_c x_1)^4 x_2 = [x_{4\alpha_1+3\alpha_2}, x_{12}]_c = 0.$$

(iii) ${}_o\zeta^8 \xrightarrow{\zeta^3} {}_o^{-1}$. The set of positive roots is the same as in Example 2.7,(ii). Following Theorem 1.6, $\mathcal{B}(V)$ has a presentation by generators x_1, x_2 , and relations

$$x_1^5 = x_2^2 = x_{3\alpha_1+\alpha_2}^{20} = x_{\alpha_1+\alpha_2}^{20} = [x_1, x_{3\alpha_1+2\alpha_2}]_c + \frac{(1 - \zeta^{17})q_{12}}{1 - \zeta^2} x_{2\alpha_1+\alpha_2}^2 = 0.$$

(iv) ${}_o\zeta^8 \xrightarrow{\zeta^{13}} {}_o^{-1}$. The set of positive roots is again the same as in Example 2.7,(ii). According to Theorem 1.6, $\mathcal{B}(V)$ has a presentation by generators x_1, x_2 , and relations

$$x_1^5 = x_2^2 = x_{3\alpha_1+\alpha_2}^{20} = x_{\alpha_1+\alpha_2}^{20} = [x_1, x_{3\alpha_1+2\alpha_2}]_c + \frac{(1 - \zeta^7)q_{12}}{1 - \zeta^2} x_{2\alpha_1+\alpha_2}^2 = 0.$$

Example 2.9. Row 15. Let $\zeta \in \mathbb{G}'_{15}$ be primitive. Notice that the roots in $\mathcal{O}(\chi)$ are those such that $N_\alpha = 30$. For each one of these vector spaces V , we have $\dim \mathcal{B}(V) = 2^4 3^4 5^4 = 30^4$.

(i) ${}_o^{-\zeta} \xrightarrow{-\zeta^{12}} {}_o\zeta^5$. The set of positive roots is

$$\{\alpha_1, 3\alpha_1 + \alpha_2, 5\alpha_1 + 2\alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, \alpha_2\}.$$

Considering the associated hyperwords, Theorem 1.6 says that $\mathcal{B}(V)$ has a presentation by generators x_1, x_2 , and relations

$$\begin{aligned} x_1^{30} &= x_2^3 = x_{3\alpha_1+\alpha_2}^{30} = (\text{ad}_c x_1)^4 x_2 = 0, \\ [x_1, x_{\alpha_1+2\alpha_2}]_c + \frac{\zeta^{10}(1+\zeta^{13})q_{12}}{1+\zeta^{12}} x_{\alpha_1+\alpha_2}^2 &= [x_{3\alpha_1+2\alpha_2}, x_{12}]_c = 0. \end{aligned}$$

(ii) ${}_o\zeta^3 \xrightarrow{-\zeta^4} {}_o^{-\zeta^{11}}$. The set of positive roots is in this case:

$$\{\alpha_1, 4\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2, 4\alpha_1 + 3\alpha_2, \alpha_1 + \alpha_2, \alpha_2\}.$$

Following Theorem 1.6, $\mathcal{B}(V)$ has a presentation by generators x_1, x_2 , and relations

$$\begin{aligned} x_1^5 &= x_2^{30} = x_{2\alpha_1+\alpha_2}^{30} = (\text{ad}_c x_2)^2 x_1 = 0, \\ [x_1, x_{3\alpha_1+2\alpha_2}]_c - \frac{(1-\zeta^2)\zeta^9 q_{12}}{1+\zeta^7} x_{2\alpha_1+\alpha_2}^2 &= [x_{4\alpha_1+3\alpha_2}, x_{12}]_c = 0. \end{aligned}$$

(iii) ${}_o\zeta^5 \xrightarrow{-\zeta^{13}} {}_o^{-1}$. The positive roots are in this case

$$\{\alpha_1, 2\alpha_1 + \alpha_2, 5\alpha_1 + 3\alpha_2, 8\alpha_1 + 5\alpha_2, 3\alpha_1 + 2\alpha_2, 4\alpha_1 + 3\alpha_2, \alpha_1 + \alpha_2, \alpha_2\}.$$

Therefore $\mathcal{B}(V)$ has a presentation by generators x_1, x_2 , and relations

$$x_1^3 = x_2^2 = x_{2\alpha_1+\alpha_2}^{30} = x_{4\alpha_1+3\alpha_2}^{30} = [x_{4\alpha_1+3\alpha_2}, x_{12}]_c = 0.$$

(iv) ${}_o\zeta^3 \xrightarrow{-\zeta^2} {}_o^{-1}$. The set of positive roots is in this case:

$$\{\alpha_1, 4\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 8\alpha_1 + 3\alpha_2, 5\alpha_1 + 2\alpha_2, 2\alpha_1 + \alpha_2, \alpha_1 + \alpha_2, \alpha_2\}.$$

Therefore $\mathcal{B}(V)$ has a presentation by generators x_1, x_2 , and relations

$$x_1^5 = x_2^2 = x_{4\alpha_1+\alpha_2}^{30} = x_{2\alpha_1+\alpha_2}^{30} = [x_{2\alpha_1+\alpha_2}, x_{12}]_c = [x_{5\alpha_1+2\alpha_2}, x_{112}]_c = 0.$$

Example 2.10. Row 16. Let $\zeta \in \mathbb{G}'_7$ be primitive. The roots in $\mathcal{O}(\chi)$ are those such that $q_\alpha \in \{-\zeta, -\zeta^5\}$. For each one of these vector spaces V , we have $\dim \mathcal{B}(V) = 2^{12}7^6$.

(i) ${}_o\zeta \xrightarrow{\zeta^2} {}_o^{-1}$. The set of positive roots is in this case:

$$\begin{aligned} \Delta_+^\chi &= \{\alpha_1, 3\alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 7\alpha_1 + 4\alpha_2, 5\alpha_1 + 3\alpha_2, 8\alpha_1 + 5\alpha_2, \\ &\quad 3\alpha_1 + 2\alpha_2, 7\alpha_1 + 5\alpha_2, 4\alpha_1 + 3\alpha_2, 5\alpha_1 + 4\alpha_2, \alpha_1 + \alpha_2, \alpha_2\}. \end{aligned}$$

Following Theorem 1.6, $\mathcal{B}(V)$ has a presentation by generators x_1, x_2 , and relations

$$\begin{aligned} x_2^2 &= x_\alpha^{14} = 0, \quad \alpha = \alpha_1, 2\alpha_1 + \alpha_2, 5\alpha_1 + 3\alpha_2, 3\alpha_1 + 2\alpha_2, 4\alpha_1 + 3\alpha_2, \alpha_1 + \alpha_2, \\ (\text{ad}_c x_1)^4 x_2 &= [x_{2\alpha_i+\alpha_j}, x_{4\alpha_i+3\alpha_j}]_c - \frac{(1-\zeta+3\zeta^4-\zeta^6)q_{ij}}{2\zeta+\zeta^2-\zeta^3-\zeta^4+\zeta^5-2} x_{3\alpha_i+2\alpha_j}^2 = 0, \end{aligned}$$

(ii) ${}_o^{-\zeta^3} \xrightarrow{\zeta^3} {}_o^{-1}$. For this braiding, the positive roots are

$$\begin{aligned} \Delta_+^\chi &= \{\alpha_1, 5\alpha_1 + \alpha_2, 4\alpha_1 + \alpha_2, 7\alpha_1 + 2\alpha_2, 3\alpha_1 + \alpha_2, 8\alpha_1 + 3\alpha_2, \\ &\quad 5\alpha_1 + 2\alpha_2, 7\alpha_1 + 3\alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2, \alpha_1 + \alpha_2, \alpha_2\}. \end{aligned}$$

Therefore $\mathcal{B}(V)$ has a presentation by generators x_1, x_2 , and relations

$$\begin{aligned} x_2^2 &= x_\alpha^{14} = 0, \quad \alpha = \alpha_1, 4\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 5\alpha_1 + 2\alpha_2, 2\alpha_1 + \alpha_2, \alpha_1 + \alpha_2, \\ (\text{ad}_c x_1)^6 x_2 &= [x_1, x_{3\alpha_1+2\alpha_2}]_c + \frac{(1+\zeta^4)q_{12}}{1-\zeta^2} x_{2\alpha_1+\alpha_2}^2 = 0, \end{aligned}$$

3. EXAMPLES IN RANK 3 AND 4

Now we consider three Weyl equivalences classes in rank three and two in rank four, and make the same work as in the previous Section.

Example 3.1. Rank 3, row 13. In this case we have two different Weyl groupoids, depending on the order of ζ , with the same root systems (the associated braidings are different). We analyze each case.

1. Let $\zeta \in \mathbb{G}'_3$ be primitive. The roots in $\mathcal{O}(\chi)$ are those such that $\text{ord } q_\alpha \in \{3, 6\}$. For each one of these vector spaces V , $\dim \mathcal{B}(V) = 2^4 3^6 6^3 = 2^7 3^9$.

(i) $\circ \zeta \xrightarrow{\zeta^2} \circ \zeta \xrightarrow{\zeta} \circ^{-1}$. The set of positive roots is the following:

$$\Delta_+^\chi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, 2\alpha_1 + 2\alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + 2\alpha_2 + \alpha_3, \\ \alpha_1 + 2\alpha_2 + 2\alpha_3, 2\alpha_2 + \alpha_3, \alpha_1 + 3\alpha_2 + 2\alpha_3, 2\alpha_1 + 3\alpha_2 + 2\alpha_3, 2\alpha_1 + 4\alpha_2 + 3\alpha_3\}.$$

According to Theorem 1.6, $\mathcal{B}(V)$ has a presentation by generators x_1, x_2, x_3 , and relations

$$x_3^2 = (\text{ad}_c x_1)^2 x_2 = (\text{ad}_c x_1) x_3 = (\text{ad}_c x_2)^2 x_1 = [x_{223}, x_{23}]_c = [[x_{123}, x_2]_c, x_2]_c = 0, \\ x_\alpha^3 = 0, \quad \alpha = \alpha_1, \alpha_1 + \alpha_2, \alpha_2, \alpha_1 + 2\alpha_2 + 2\alpha_3, \alpha_1 + 3\alpha_2 + 2\alpha_3, 2\alpha_1 + 3\alpha_2 + 2\alpha_3, \\ x_\alpha^6 = 0, \quad \alpha = \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + 2\alpha_2 + \alpha_3.$$

(ii) $\circ \zeta \xrightarrow{\zeta^2} \circ \xrightarrow{\zeta^2} \circ^{-1}$. In this case, we have

$$\Delta_+^\chi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, 2\alpha_1 + 2\alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3, 2\alpha_1 + 4\alpha_2 + \alpha_3, \\ 2\alpha_2 + \alpha_3, \alpha_2 + \alpha_3, \alpha_1 + 3\alpha_2 + \alpha_3, 2\alpha_1 + 3\alpha_2 + \alpha_3, \alpha_1 + 2\alpha_2 + \alpha_3\}.$$

Theorem 1.6 says that $\mathcal{B}(V)$ is presented by generators x_1, x_2, x_3 , and relations

$$x_3^2 = (\text{ad}_c x_1)^2 x_2 = (\text{ad}_c x_1) x_3 = (\text{ad}_c x_2)^3 x_1 = (\text{ad}_c x_2)^3 x_3 = 0, \\ [x_3, x_{221}]_c + q_{21}[x_{321}, x_2]_c + q_{32}\zeta^2(1 - \zeta^2)x_2 x_{321} = 0, \\ x_\alpha^3 = 0, \quad \alpha = \alpha_1, \alpha_1 + \alpha_2 + \alpha_3, \alpha_2 + \alpha_3, \alpha_1 + 2\alpha_2, \alpha_1 + 3\alpha_2 + \alpha_3, 2\alpha_1 + 3\alpha_2 + \alpha_3, \\ x_\alpha^6 = 0, \quad \alpha = \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2 + \alpha_3.$$

2. Now, let $\zeta \in \mathbb{G}'_6$ be primitive. The roots in $\mathcal{O}(\chi)$ are those such that $\text{ord } q_\alpha = 6$. In this case, $\dim \mathcal{B}(V) = 2^4 3^3 6^6 = 2^{10} 3^9$.

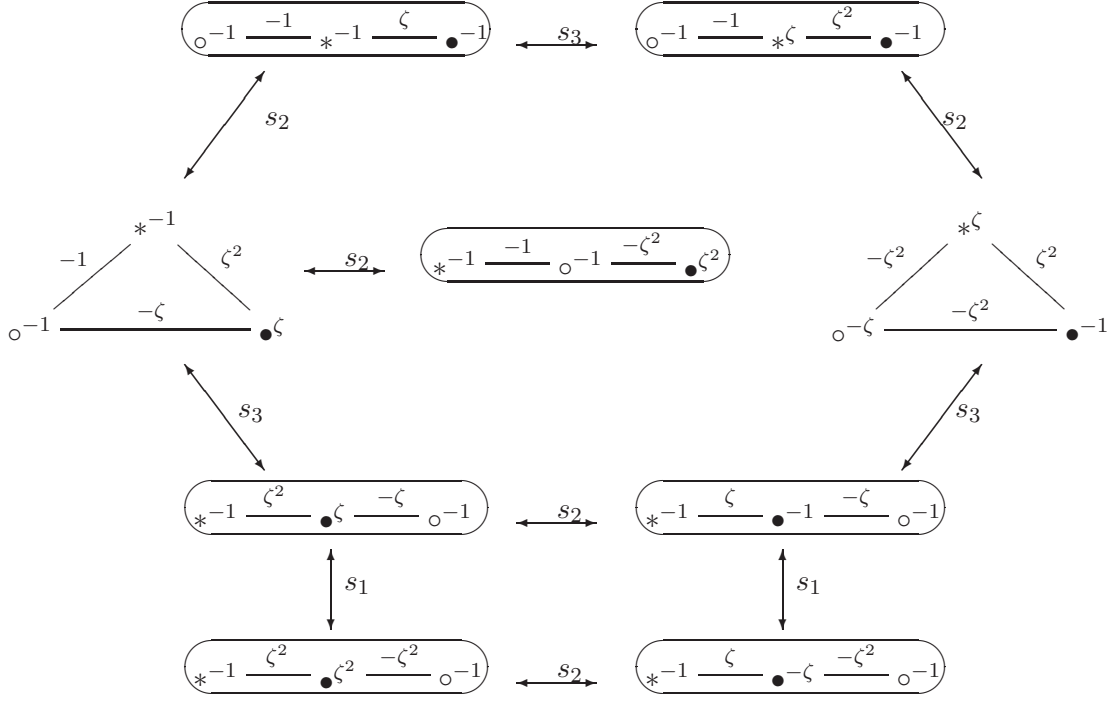
(i) $\circ \zeta \xrightarrow{\zeta^5} \circ \zeta \xrightarrow{\zeta^4} \circ^{-1}$. The set of positive roots is the same as in **1.(i)**. Using Theorem 1.6, we deduce that $\mathcal{B}(V)$ has a presentation by generators x_1, x_2, x_3 , and relations

$$x_3^2 = (\text{ad}_c x_1)^2 x_2 = (\text{ad}_c x_1) x_3 = (\text{ad}_c x_2)^2 x_1 = (\text{ad}_c x_2)^3 x_2 = 0, \\ x_\alpha^6 = 0, \quad \alpha = \alpha_1, \alpha_1 + \alpha_2, \alpha_2, \alpha_1 + 2\alpha_2 + 2\alpha_3, \alpha_1 + 3\alpha_2 + 2\alpha_3, 2\alpha_1 + 3\alpha_2 + 2\alpha_3.$$

(ii) $\circ \zeta \xrightarrow{\zeta^2} \circ \xrightarrow{\zeta^2} \circ^{-1}$. The positive roots are the same as in **1.(ii)**. By Theorem 1.6, $\mathcal{B}(V)$ is presented by generators x_1, x_2, x_3 , and relations

$$x_3^2 = x_2^3 = (\text{ad}_c x_1)^2 x_2 = (\text{ad}_c x_1) x_3 = [x_{223}, x_{23}]_c = 0, \\ [x_1, x_{223}]_c + q_{23}[x_{123}, x_2]_c - q_{12}x_2 x_{123} = 0, \\ x_\alpha^6 = 0, \quad \alpha = \alpha_1, \alpha_1 + \alpha_2 + \alpha_3, \alpha_2 + \alpha_3, \alpha_1 + 2\alpha_2, \alpha_1 + 3\alpha_2 + \alpha_3, 2\alpha_1 + 3\alpha_2 + \alpha_3.$$

Example 3.2. Rank 3, row 17. Let $\zeta \in \mathbb{G}'_3$ be primitive. The following diagram shows the action of the Weyl groupoid, where we indicate the vertices 1, 2, 3 by \circ , $*$, \bullet , respectively, and we omit those symmetries which do not change the Dynkin diagram.



In this case, the roots in $\mathcal{O}(\chi)$ are those such that $N_\alpha = 6$, and $\dim \mathcal{B}(V) = 2^7 3^3 6 = 2^8 3^4$.

(i) $\circ^{-1} \xrightarrow{-1} *^{-1} \xrightarrow{\zeta} \bullet^{-1}$. We have that

$$\Delta_+^\chi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + 2\alpha_2 + \alpha_3, \alpha_1 + 2\alpha_2 + 2\alpha_3, \\ \alpha_1 + 3\alpha_2 + 2\alpha_3, 2\alpha_1 + 3\alpha_2 + 2\alpha_3, 2\alpha_1 + 4\alpha_2 + 3\alpha_3\}.$$

Following Theorem 1.6, $\mathcal{B}(V)$ has a presentation by generators x_1, x_2, x_3 , and relations

$$x_1^2 = x_2^2 = x_3^2 = x_{12}^2 = (\text{ad}_c x_1)x_3 = [x_{32}, [x_{32}, x_{321}]_c]_c = x_{\alpha_1 + 2\alpha_2 + 2\alpha_3}^6 = 0.$$

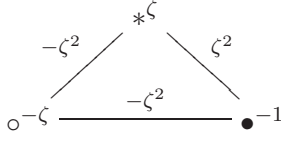
(ii) $\circ^{-1} \xrightarrow{-1} *^\zeta \xrightarrow{\zeta^2} \bullet^{-1}$. For this braiding,

$$\Delta_+^\chi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + 3\alpha_2 + \alpha_3, \\ 2\alpha_1 + 3\alpha_2 + \alpha_3, 2\alpha_1 + 4\alpha_2 + \alpha_3, \alpha_1 + 2\alpha_2 + \alpha_3\}.$$

By Theorem 1.6, $\mathcal{B}(V)$ has a presentation by generators x_1, x_2, x_3 , and relations

$$x_1^2 = x_2^3 = x_3^2 = (\text{ad}_c x_2)^2 x_3 = (\text{ad}_c x_1)x_3 = [x_{221}, x_{21}]_c = [x_{12}, x_{123}]_c = x_{\alpha_1 + 2\alpha_2}^6 = 0.$$

(iii) . The root system is



$$\Delta_+^\chi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + 2\alpha_2, \alpha_1 + \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, 2\alpha_1 + \alpha_2 + \alpha_3, \\ \alpha_1 + 2\alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3, 2\alpha_1 + 2\alpha_2 + \alpha_3\}.$$

By Theorem 1.6, $\mathcal{B}(V)$ admits a presentation by generators x_1, x_2, x_3 , and relations

$$x_1^6 = x_2^3 = x_3^2 = (\text{ad}_c x_2)^2 x_3 = (\text{ad}_c x_1)^2 x_2 = (\text{ad}_c x_1)^2 x_3 = 0 \\ x_{123} = -q_{23}(1 - \zeta^2)[x_{13}, x_2]_c + q_{12}(1 - \zeta^2)x_2 x_{13}.$$

(iv) $*^{-1} \xrightarrow{\zeta} \bullet^{-1} \xrightarrow{-\zeta} o^{-1}$. In this case,

$$\Delta_+^\chi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_2 + \alpha_3, \alpha_3 + \alpha_1, \alpha_1 + 2\alpha_2 + 2\alpha_3, \alpha_1 + \alpha_2 + \alpha_3, 2\alpha_2 + 3\alpha_3 + \alpha_1, \\ \alpha_2 + 2\alpha_3 + \alpha_1, \alpha_2 + 2\alpha_3 + 2\alpha_1, 2\alpha_2 + 3\alpha_3 + 2\alpha_1\}.$$

Following Theorem 1.6, $\mathcal{B}(V)$ has a presentation by generators x_1, x_2, x_3 , and relations

$$x_1^2 = x_2^2 = x_3^2 = (\text{ad}_c x_1)x_2 = [[x_{13}, x_{132}]_c, x_3]_c = x_{\alpha_1 + \alpha_3}^6 = 0.$$

(v) $*^{-1} \xrightarrow{\zeta} \bullet^{-\zeta} \xrightarrow{-\zeta^2} o^{-1}$. Its root system is

$$\Delta_+^\chi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_2 + \alpha_3, \alpha_2 + 2\alpha_3, \alpha_3 + \alpha_1, 2\alpha_2 + 2\alpha_3 + \alpha_1, \alpha_2 + \alpha_3 + \alpha_1, \\ 2\alpha_2 + 3\alpha_3 + \alpha_1, \alpha_2 + 2\alpha_3 + \alpha_1, 2\alpha_2 + 3\alpha_3 + 2\alpha_1\}.$$

Following Theorem 1.6, $\mathcal{B}(V)$ has a presentation by generators x_1, x_2, x_3 , and relations

$$x_1^2 = x_2^2 = x_3^6 = (\text{ad}_c x_3)^3 x_2 = (\text{ad}_c x_3)^2 x_1 = (\text{ad}_c x_1)x_2 = 0.$$

(vi) $*^{-1} \xrightarrow{\zeta^2} \bullet^{\zeta^2} \xrightarrow{-\zeta^2} o^{-1}$. The corresponding root system is

$$\Delta_+^\chi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_2 + \alpha_3, \alpha_2 + 2\alpha_3, 2\alpha_3 + \alpha_1, \alpha_3 + \alpha_1, \alpha_2 + \alpha_3 + \alpha_1, \\ \alpha_2 + 3\alpha_3 + \alpha_1, \alpha_2 + 3\alpha_3 + 2\alpha_1, \alpha_2 + 2\alpha_3 + \alpha_1\}.$$

By Theorem 1.6, it follows that $\mathcal{B}(V)$ is presented by generators x_1, x_2, x_3 , and relations

$$x_1^2 = x_2^2 = x_3^3 = x_{\alpha_2 + \alpha_3}^6 = (\text{ad}_c x_1)x_2 = [x_{331}, x_{31}]_c = 0 \\ [x_1, x_{332}]_c = -q_{23}^{-1}\zeta^2[x_{132}, x_3]_c + q_{13}x_3x_{132}.$$

(vii) $*^{-1} \xrightarrow{\zeta^2} \bullet^{\zeta} \xrightarrow{-\zeta} o^{-1}$. We have the following positive roots:

$$\Delta_+^\chi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_2 + \alpha_3, 2\alpha_3 + \alpha_1, \alpha_3 + \alpha_1, \alpha_2 + \alpha_3 + \alpha_1, \alpha_2 + 3\alpha_3 + \alpha_1, \\ \alpha_2 + 2\alpha_3 + 2\alpha_1, \alpha_2 + 2\alpha_3 + \alpha_1, \alpha_2 + 3\alpha_3 + 2\alpha_1\}.$$

Theorem 1.6 implies that $\mathcal{B}(V)$ has a presentation by generators x_1, x_2, x_3 , and relations

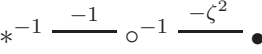
$$x_1^2 = x_2^2 = x_3^3 = x_{\alpha_2 + \alpha_3 + \alpha_1}^6 = (\text{ad}_c x_3)^2 x_2 = (\text{ad}_c x_1)x_2 = [x_{331}, x_{31}]_c = 0.$$

(viii)  . For this braiding,

$$\Delta_+^\chi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_2 + \alpha_1, \alpha_2 + \alpha_3, \alpha_1 + \alpha_3, \alpha_1 + 2\alpha_3, \alpha_2 + \alpha_1 + \alpha_3, \\ \alpha_2 + \alpha_1 + 2\alpha_3, \alpha_2 + 2\alpha_1 + 2\alpha_3, \alpha_2 + 2\alpha_1 + 3\alpha_3\}.$$

Following Theorem 1.6, $\mathcal{B}(V)$ has a presentation by generators x_1, x_2, x_3 , and relations

$$x_1^2 = x_2^2 = x_3^3 = x_{12}^2 = x_{\alpha_2 + \alpha_1 + 2\alpha_3}^6 = (\text{ad}_c x_3)^3 x_2 = [x_{331}, x_{31}]_c = 0 \\ x_{123} = q_{23}(\zeta - \zeta^2)[x_{13}, x_2]_c + q_{12}(1 - \zeta)x_2 x_{13}.$$

(ix)  . Its positive roots are

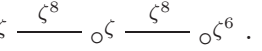
$$\Delta_+^\chi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_2 + \alpha_1, \alpha_1 + \alpha_3, \alpha_1 + 2\alpha_3, \alpha_2 + \alpha_1 + \alpha_3, \alpha_2 + \alpha_1 + 2\alpha_3, \\ \alpha_2 + 2\alpha_1 + \alpha_3, \alpha_2 + 2\alpha_1 + 3\alpha_3, \alpha_2 + 2\alpha_1 + 2\alpha_3\}.$$

According to Theorem 1.6, $\mathcal{B}(V)$ is presented by generators x_1, x_2, x_3 , and relations

$$x_1^2 = x_2^2 = x_3^3 = x_{\alpha_2 + 2\alpha_1 + 2\alpha_3}^6 = (\text{ad}_c x_2)x_3 = [x_{331}, x_{31}]_c = [x_{3312}, x_{312}]_c = 0 \\ x_{12}^2 = [[x_{31}, x_{312}]_c, x_1]_c = 0.$$

Example 3.3. Rank 3, row 18. Let $\zeta \in \mathbb{G}'_9$ be primitive. We distinguish this Weyl equivalence class of root systems because all the other unidentified cases in rank 3 have elements of \mathbb{G}'_6 labelling their vertices and edges.

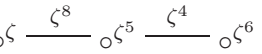
In this case, the roots in $\mathcal{O}(\chi)$ are those such that $N_\alpha = 9$, and $\dim \mathcal{B}(V) = 9^9 3^4 = 3^{22}$.

(i)  . We have that

$$\Delta_+^\chi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + 2\alpha_3, \alpha_1 + 2\alpha_2 + 3\alpha_3, \\ \alpha_2 + 2\alpha_3, \alpha_1 + 2\alpha_2 + 4\alpha_3, \alpha_1 + 3\alpha_2 + 4\alpha_3, \alpha_1 + 2\alpha_2 + 2\alpha_3, 2\alpha_1 + 3\alpha_2 + 4\alpha_3\}.$$

Following Theorem 1.6, $\mathcal{B}(V)$ has a presentation by generators x_1, x_2, x_3 , and relations

$$x_3^3 = (\text{ad}_c x_1)^2 x_2 = (\text{ad}_c x_1)x_3 = (\text{ad}_c x_2)^2 x_1 = (\text{ad}_c x_2)^2 x_3 = 0; \\ x_\alpha^9 = 0, \quad \alpha \neq \alpha_3, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + 2\alpha_2 + 3\alpha_3; \\ [[x_{332}, x_{3321}]_c, x_{32}]_c = 0.$$

(ii)  . For this braiding,

$$\Delta_+^\chi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + 2\alpha_2, \alpha_2 + 2\alpha_3, \alpha_1 + \alpha_2 + 2\alpha_3, \\ \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + 2\alpha_2 + \alpha_3, \alpha_1 + 3\alpha_2 + 2\alpha_3, \alpha_1 + 2\alpha_2 + 2\alpha_3, 2\alpha_1 + 3\alpha_2 + 2\alpha_3\}.$$

Therefore $\mathcal{B}(V)$ has a presentation by generators x_1, x_2, x_3 , and relations

$$x_3^3 = (\text{ad}_c x_1)^2 x_2 = (\text{ad}_c x_1)x_3 = (\text{ad}_c x_2)^3 x_1 = (\text{ad}_c x_2)^2 x_3 = 0; \\ x_\alpha^9 = 0, \quad \alpha \neq \alpha_3, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + 2\alpha_2 + \alpha_3; \\ (1 + \zeta^4)[[x_{123}, x_2]_c, x_3]_c = q_{jk}[[x_{123}, x_3]_c, x_2]_c.$$

Example 3.5. Rank 4, row 22. Let $\zeta \in \mathbb{G}'_4$. This Weyl equivalence class contains eight different diagrams:

$$\begin{array}{ll}
 \text{(i)} \quad \circ^{-\zeta} \xrightarrow{\zeta} \circ^{-1} \xrightarrow{-\zeta} \circ \xrightarrow{\zeta} \circ^{-\zeta} & \text{(ii)} \quad \circ^{-1} \xrightarrow{-\zeta} \circ^{-1} \xrightarrow{\zeta} \circ^{-1} \xrightarrow{\zeta} \circ^{-\zeta} \\
 \text{(iii)} \quad \circ^{-\zeta} \xrightarrow{\zeta} \circ^{-1} \xrightarrow{-\zeta} \circ \xrightarrow{\zeta} & \text{(iv)} \quad \circ^{-1} \xrightarrow{-\zeta} \circ \xrightarrow{\zeta} \circ^{-1} \xrightarrow{-\zeta} \circ^{-1} \\
 \quad \quad \quad \begin{array}{c} -1 \mid \nearrow \\ \circ^{-1} \quad -\zeta \end{array} & \quad \quad \begin{array}{c} -1 \mid \nearrow \\ \circ^{-1} \quad -\zeta \end{array} \\
 \text{(v)} \quad \circ^{-1} \xrightarrow{\zeta} \circ^{-\zeta} \xrightarrow{\zeta} \circ^{-1} \xrightarrow{\zeta} \circ^{-\zeta} & \text{(vi)} \quad \circ^{-1} \xrightarrow{\zeta} \circ^{-1} \xrightarrow{-\zeta} \circ^{-1} \xrightarrow{-\zeta} \circ^{-1} \\
 \quad \quad \quad \begin{array}{c} -1 \mid \nearrow \\ \circ^{-1} \quad -\zeta \end{array} & \quad \quad \begin{array}{c} -1 \mid \nearrow \\ \circ^{-1} \quad -\zeta \end{array} \\
 \text{(vii)} \quad \circ^{-1} \xrightarrow{-\zeta} \circ \xrightarrow{-1} \circ^{-1} \xrightarrow{\zeta} \circ^{-\zeta} & \text{(viii)} \quad \circ^{-1} \xrightarrow{\zeta} \circ^{-1} \xrightarrow{-1} \circ^{-1} \xrightarrow{\zeta} \circ^{-\zeta}
 \end{array}$$

Each one of the associated Nichols algebras has dimension 2^{42} . They are presented by generators x_1, x_2, x_3, x_4 , and relations:

- if $N_\alpha = 4$: $x_\alpha^{N_\alpha}$;
- if $q_{ii} = -1$: x_i^2 ;
- if $q_{ii} = q_{jj} = \widetilde{q}_{ij} = -1$ (diagrams (iii), (vi), (viii)): x_{ij}^2 ;
- if $\widetilde{q}_{ij} = 1$: $(\text{ad}_c x_i)x_j$;
- if $\widetilde{q}_{ij} = q_{ii}^{-1} = \pm\zeta$: $(\text{ad}_c x_i)^2 x_j$;
- if $\widetilde{q}_{ij} = -1$, $q_{ii} = \zeta$ (diagrams (iv), (vii)): $[x_{ij}, x_{ij}]_c$;
- if $\widetilde{q}_{ij} = \widetilde{q}_{kj} = -\zeta$, $\widetilde{q}_{ik} = -1$ (diagrams (iii), (iv), (vi)): $x_{ijk} - \zeta q_{jk}[x_{ik}, x_j]_c - q_{ij}(1 + \zeta)x_j x_{ik}$;
- if $-q_{jj} = \widetilde{q}_{ij}\widetilde{q}_{kj} = \widetilde{q}_{ik} = 1$ (diagrams (i), (ii), (iii), (vi)): $[x_{ijk}, x_j]_c$;
- if $q_{jj} = \widetilde{q}_{ij} = -\widetilde{q}_{kj} = \zeta$, $\widetilde{q}_{ik} = 1$ (diagram (i)): $[[x_{ijk}, x_j]_c, x_j]_c$;
- if $q_{ii} = q_{jj} - 1$, $-\widetilde{q}_{ij} = \widetilde{q}_{kj} = \zeta$, $\widetilde{q}_{ik} = 1$ (diagram (ii)): $[x_{ij}, [x_{ij}, x_{ijk}]_c]_c, x_j]_c$;
- for diagram (v): $[x_{123}, [x_{1234}, x_3]_c, x_{23}]_c$.

4. THE OTHER UNIDENTIFIED NICHOLS ALGEBRAS

In this Section we will consider the remaining braidings of unidentified type. We will consider four different subfamilies, according with the shape of their generalized Dynkin diagrams. The first subfamily that we will study is closely related with diagrams of type D_5, E_6, E_7 , but they contain non-Cartan vertices labeled with -1 and small orders on the q_{ii} (in fact, they are roots of unity of order 3 or 4).

Theorem 4.1. *Let (V, c) a braided vector space of diagonal type of dimension $\theta \in \{5, 6, 7\}$, whose generalized Dynkin diagram belongs to rows 11, 14, 17, 19 or 21 of [H2, Table 4]. Then $\mathcal{B}(V)$ is presented by generators x_1, \dots, x_θ and relations*

- if $N_\alpha \neq 2$ or $N_\alpha = 2$, $\alpha = \alpha_i$: $x_\alpha^{N_\alpha}$;
- if i, j are such that $q_{ii} = \widetilde{q}_{ij} = q_{jj} = -1$: x_{ij}^2 ;
- if i, j are such that $\widetilde{q}_{ij} = 1$ (respectively, $\widetilde{q}_{ij} = q_{ii}^{-1} \neq -1$): $(\text{ad}_c x_i)x_j$ (resp. $(\text{ad}_c x_i)^2 x_j$);
- if i, j, k are such that $\widetilde{q}_{ik} = 1$, $q_{jj} = -1$, $\widetilde{q}_{ij}\widetilde{q}_{jk} = 1$: $[x_{ijk}, x_j]_c$;

- ⊙ if i, j, k are such that $\widetilde{q}_{ik} = \widetilde{q}_{ij} = \widetilde{q}_{jk} =: \zeta \in \mathbb{G}'_3$:

$$x_{ijk} - q_{jk}\zeta^2[x_{ik}, x_j]_c - q_{ij}(1 - \zeta)x_jx_{ik};$$

- ⊙ if i, j, k are such that $\widetilde{q}_{ik} = \widetilde{q}_{jk} =: \zeta \in \mathbb{G}'_4$, $\widetilde{q}_{ij} = -1$:

$$x_{ijk} + q_{jk}\zeta[x_{ik}, x_j]_c - q_{ij}(1 - \zeta)x_jx_{ik}.$$

Proof. First, note that $a_{ij} \in \{0, -1\}$ for all these braidings; that is, for each pair $i \neq j$, either $\widetilde{q}_{ij} = 1$, or else $\widetilde{q}_{ij} \neq -1$, $q_{ii} \in \{-1, \widetilde{q}_{ij}^{-1}\}$. This explains why we only consider quantum Serre relations associated to $a_{ij} = 0, 1$. We derive also that the Cartan vertices are those labeled with $q_{ii} \neq -1$, which explains why we need only the power root vectors associated to simple roots, or to other roots such that $N_\alpha \neq 2$.

Finally we look at the other needed relations. As the a_{ij} 's take only two values, we need few extra relations. \square

The second subfamily seems close to type super $D(n)$, but with certain 'degeneration' and small order on the labels of the vertices (roots of unity of order 2, 3, 5). With these diagrams and the corresponding to the previous family we cover all the cases in rank 5, 6, 7.

Theorem 4.2. *Let (V, c) a braided vector space of diagonal type of dimension $\theta \in \{3, 4, 5, 6\}$, whose generalized Dynkin diagram belongs to row 15 of [H2, Table 2], or row 18 of [H2, Table 3], or rows 12, 13, 15 or 18 of [H2, Table 4]. Then $\mathcal{B}(V)$ is presented by generators x_1, \dots, x_θ and relations*

- ⊙ if $N_\alpha \neq 2$: $x_\alpha^{N_\alpha}$;
- ⊙ if $q_{ii} = -1$: x_i^2 ;
- ⊙ if i, j are such that $\widetilde{q}_{ij} = 1$: $(\text{ad}_c x_i)x_j$;
- ⊙ if i, j are such that $\widetilde{q}_{ij} = q_{ii}^{-1} \neq -1$: $(\text{ad}_c x_i)^2 x_j$;
- ⊙ if i, j are such that $\widetilde{q}_{ij} = q_{ii}^{-2} \in \mathbb{G}'_5$: $(\text{ad}_c x_i)^3 x_j$;
- ⊙ if i, j, k are such that $\widetilde{q}_{ik} = 1$, $q_{jj} = -1$, $\widetilde{q}_{ij}\widetilde{q}_{jk} = 1$: $[x_{ijk}, x_j]_c$;
- ⊙ if i, j, k are such that $\widetilde{q}_{ik} = \widetilde{q}_{ij} = \widetilde{q}_{jk} =: \zeta \in \mathbb{G}'_3$:

$$x_{ijk} - q_{jk}\zeta^2[x_{ik}, x_j]_c - q_{ij}(1 - \zeta)x_jx_{ik};$$

- ⊙ if i, j, k are such that $\widetilde{q}_{ij} =: \zeta \in \mathbb{G}'_5$, $\widetilde{q}_{ik} = \widetilde{q}_{jk} = \zeta^2$:

$$x_{ijk} - q_{jk}\zeta^3[x_{ik}, x_j]_c - q_{ij}(1 - \zeta^2)x_jx_{ik}.$$

- ⊙ if i, j, k are such that $\widetilde{q}_{ik} = 1$, $q_{jj} = \widetilde{q}_{ij} = \widetilde{q}_{jk}^2 \in \mathbb{G}'_3$:

$$[[x_{ijk}, x_j]_c, x_j]_c;$$

- ⊙ if i, j, k are such that $\widetilde{q}_{ik} = 1$, $q_{ii} = q_{jj} = -1$, $\widetilde{q}_{jk} = \widetilde{q}_{ik}^{-2}$:

$$[[x_{ij}, x_{ijk}]_c, x_j]_c.$$

Proof. For these braidings, $a_{ij} \in \{0, -1, -2\}$. Moreover, the Cartan vertices are those such that $q_{ii} \neq -1$. In fact, when $q_{ii} \in \mathbb{G}'_3$ and there exists j such that $a_{ij} = -2$, then $q_{ij} = q_{ii}$, so i is a Cartan vertex (but we do not need the corresponding quantum Serre relation). Therefore we need only the power root vectors associated to a simple root, or to other roots such that $N_\alpha \neq 2$.

The remaining relations we need to generate the ideal expresses the similarity with the super $D(n)$ case. \square

The following subfamily keeps certain similarity with diagrams of type super $F(4)$ for small orders on the labels of the vertices (3, 6).

Theorem 4.3. *Let (V, c) a braided vector space of diagonal type of dimension $\theta = 4$, whose generalized Dynkin diagram belongs to rows 20 or 21 of [H2, Table 3]. Then $\mathcal{B}(V)$ is presented by generators x_1, x_2, x_3, x_4 and relations*

- ⊙ if $N_\alpha = 3, 6$: $x_\alpha^{N_\alpha}$;
- ⊙ if $q_{ii} = -1$: x_i^2 ;
- ⊙ if i, j are such that $\widetilde{q}_{ij} = 1$: $(\text{ad}_c x_i)x_j$;
- ⊙ if i, j are such that $\widetilde{q}_{ij} = q_{ii}^{-1} \neq -1$: $(\text{ad}_c x_i)^2 x_j$;
- ⊙ if i, j are such that $\widetilde{q}_{ij} = q_{ii}^{-2}$, $q_{ii} \in \mathbb{G}'_6$: $(\text{ad}_c x_i)^3 x_j$;
- ⊙ if i, j, k are such that $q_{jj} = -1$, $\widetilde{q}_{ij}\widetilde{q}_{jk} = 1$, $\widetilde{q}_{ik} = 1$: $[x_{ijk}, x_j]_c$;
- ⊙ if i, j, k are such that $\widetilde{q}_{ik} = 1$, $q_{jj} = \widetilde{q}_{ij} = \widetilde{q}_{jk}^2 \in \mathbb{G}'_3$:

$$[[x_{ijk}, x_j]_c, x_j]_c$$
;
- ⊙ if i, j, k are such that $\widetilde{q}_{ik} = 1$, $q_{ii} = q_{jj} = -1$, $\widetilde{q}_{jk} = \widetilde{q}_{ik} \in \mathbb{G}'_3$:

$$[[x_{ij}, x_{ijk}]_c, x_j]_c$$
;
- ⊙ if i, j, k are such that $\widetilde{q}_{ik} = \widetilde{q}_{ij} = \widetilde{q}_{jk} =: \zeta \in \mathbb{G}'_3$:

$$x_{ijk} - q_{jk}\zeta^2[x_{ik}, x_j]_c - q_{ij}(1 - \zeta)x_jx_{ik}$$
;
- ⊙ if i, j, k are such that $\widetilde{q}_{ik} = 1$, $-q_{jj} = \widetilde{q}_{ij} = \widetilde{q}_{jk} =: \zeta \in \mathbb{G}'_3$:

$$[x_i, x_{jjk}]_c + q_{jk}[x_{ijk}, x_j]_c + q_{ij}(\zeta - \zeta^2)x_jx_{ijk}.$$

Proof. It is analogous to the proof of Theorem 4.2. □

Remark 4.4. If the diagram of (V, c) belongs to row 20 of [H2, Table 3], then

$$\dim \mathcal{B}(V) = 2^{13}3^{10}.$$

On the other hand, if it belongs to row 21 of [H2, Table 3], then

$$\dim \mathcal{B}(V) = 2^{20}3^{16}.$$

The last subfamily contains only two Weyl equivalence classes, where the labels of the vertices are roots of unity of order 2, 3, 6.

Theorem 4.5. *Let (V, c) a braided vector space of diagonal type of dimension $\theta = 3$ (resp. $\theta = 4$), whose generalized Dynkin diagram belongs to row 16 of [H2, Table 2], (resp. row 17 of [H2, Table 3]). Then $\mathcal{B}(V)$ is presented by generators x_1, \dots, x_θ and relations*

- ⊙ if $N_\alpha = 6$: x_α^6 ;
- ⊙ if $N_i = 2, 3$: $x_i^{N_i}$;
- ⊙ if i, j are such that $\widetilde{q}_{ij} = 1$: $(\text{ad}_c x_i)x_j$;
- ⊙ if i, j are such that $\widetilde{q}_{ij} = q_{ii}^{-1} \neq -1$: $(\text{ad}_c x_i)^2 x_j$;
- ⊙ if i, j are such that $\widetilde{q}_{ij} = q_{ii} = q_{jj} = -1$: x_{ij}^2 ;
- ⊙ if i, j are such that $q_{ii} \in \mathbb{G}'_3$, $q_{jj} = \widetilde{q}_{ij} = -1$: $[x_{ii}, x_{ij}]_c$;
- ⊙ if i, j, k are such that $\widetilde{q}_{ij} = -\widetilde{q}_{ik} = -\widetilde{q}_{jk} =: \zeta \in \mathbb{G}'_3$:

$$x_{ijk} + q_{jk}\zeta^2[x_{ik}, x_j]_c + q_{ij}\zeta^2 x_jx_{ik}$$
;
- ⊙ if i, j, k are such that $\widetilde{q}_{ij} = -1$, $\widetilde{q}_{ik}^{-1} = -\widetilde{q}_{jk} =: \zeta \in \mathbb{G}'_3$:

$$x_{ijk} + \frac{1}{3}q_{jk}\zeta[x_{ik}, x_j]_c + q_{ij}\zeta^2 x_jx_{ik}$$
;
- ⊙ if i, j, k, l are such that $\widetilde{q}_{ij} \in \mathbb{G}'_6$, $q_{jj}^{-1} = \widetilde{q}_{kj} = \widetilde{q}_{ij}^2$, $\widetilde{q}_{kl} = q_{kk} = -1$, $\widetilde{q}_{il} = \widetilde{q}_{ik} = \widetilde{q}_{jl} = 1$:

$$[[x_{ijk}, [x_{ijkl}, x_k]_c], x_{jk}]_c.$$

Proof. Now $a_{ij} \in \{0, -1, -2\}$, and the Cartan vertices are those such that $q_{ii}^2, q_{ii}^3 \neq 1$. The proof follows then as in the previous cases. \square

Remark 4.6. If the diagram of (V, c) belongs to [H2, Table 2, row 16], $\dim \mathcal{B}(V) = 2^7 3^6$.

If it belongs to [H2, Table 3, row 17], then $\dim \mathcal{B}(V) = 2^{19} 3^{15}$.

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